ANTI-SYMPLECTIC INVOLUTION AND FLOER COHOMOLOGY

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Abstract. The main purpose of the present paper is a study of orientations of the moduli spaces of pseudo-holomorphic discs with boundary lying on a *real* Lagrangian submanifold, i.e., the fixed point set of an anti-symplectic involutions τ on a symplectic manifold. We introduce the notion of τ-relatively spin structure for an anti-symplectic involution τ, and study how the orientations on the moduli space behave under the involution τ. We also apply this to the study of Lagrangian Floer theory of real Lagrangian submanifolds. In particular, we study unobstructedness of the τ-fixed point set of symplectic manifolds and in particular prove its unobstructedness in the case of Calabi-Yau manifolds. And we also do explicit calculation of Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,nov}^{\mathbb{Z}}$ which provides an example whose Floer cohomology is not isomorphic to its classical cohomology. We study Floer cohomology of the diagonal of the square of a symplectic manifold, which leads to a rigorous construction of the quantum Massey product of symplectic manifold in complete generality.

Contents

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1. Introduction and statement of results

An *anti-symplectic involution* τ on a symplectic manifold (M,ω) is an involution on M which satisfies $\tau^*\omega = -\omega$. Two prototypes of anti-symplectic involutions are the complex conjugation of a complex projective space with respect to the Fubini-Study metric and the canonical reflection along the zero section on the cotangent bundle. (See also [CMS] for a construction of an interesting class of anti-symplectic involutions on Lagrangian torus fibrations.) The fixed point set of τ, if it is non-empty, gives an example of Lagrangian submanifolds. For instance, the set of real points of a complex projective manifold defined over $\mathbb{R}$ belongs to this class. In this paper we study Lagrangian intersection Floer theory for the fixed point set of an anti-symplectic involution.

Let (M,ω) be a compact, or more generally tame, $2n$-dimensional symplectic manifold and L an oriented closed Lagrangian submanifold of M. It is well-known by now that the Floer cohomology of a Lagrangian submanifold L can not be defined in general. The phenomenon of bubbling-off holomorphic discs is the main source of troubles in defining Floer cohomology of Lagrangian submanifolds. In our book [FOOO3], we developed general theory of obstructions and deformations of Lagrangian intersection Floer cohomology based on the theory of filtered A_∞ algebras which we associate to each Lagrangian submanifold. However it is generally very hard to formulate the criterion for unobstructedness to defining Floer cohomology let alone to calculate Floer cohomology for a given Lagrangian submanifold. In this regard, Lagrangian torus fibers in toric manifolds provide good test cases for these problems, which we studied in [FOOO4], [FOOO5] in detail. For this class of Lagrangian submanifolds, we can do many explicit calculations of various notions and invariants that are introduced in the book [FOOO3].

Another important class of Lagrangian submanifolds is that of the fixed point set of an anti-symplectic involution. Actually, the set of real points in Calabi-Yau manifolds plays an important role of homological mirror symmetry conjecture. (See [Wa], [PSW] and [F2]. See also [We] for related topics of real points.) The purpose of the present paper is to study Floer cohomology of this class of Lagrangian submanifolds. For example, we prove unobstructedness for such Lagrangian submanifolds in Calabi-Yau manifolds and also provide some other examples of explicit calculations of Floer cohomology. The main ingredient of this paper is a careful study of orientations of the moduli spaces of pseudo-holomorphic discs.

Take an ω-compatible almost complex structure J on (M,ω). We consider moduli space $\mathcal{M}(J;\beta)$ of J-holomorphic stable maps from bordered Riemann surface $(\Sigma,\partial\Sigma)$ of genus 0 to (M,L) which represents a class $\beta \in \Pi(L) = \pi_2(M,L)/\sim$: $\beta \sim \beta' \in \pi_2(M,L)$ if and only if $\omega(\beta) = \omega(\beta')$ and $\mu_L(\beta) = \mu_L(\beta')$. Here $\mu_L : \pi_2(M,L) \to \mathbb{Z}$ is the Maslov index homomorphism. The values of μ_L are

even integers if L is oriented. When the domain Σ is a 2-disc D^2, we denote by $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ the subset of $\mathcal{M}(J;\beta)$ consisting of *smooth* maps. The moduli space $\mathcal{M}(J;\beta)$ is not orientable, in general. In Chapter 8 [FOOO3] we introduce the notion of relative spin structure on $L \subset M$ and its stable conjugacy class, and prove that if L carries a relative spin structure (V,σ), its stable conjugacy class $[(V,\sigma)]$ determines an orientation on the moduli space $\mathcal{M}(J;\beta)$ (see Sections 2, 3 for the precise definitions and notations.) We denote it by $\mathcal{M}(J;\beta)^{[(V,\sigma)]}$ when we want to specify the stable conjugacy class of the relative spin structure. If we have a diffeomorphism $f: M \to M$ satisfying $f(L) = L$, we can define the pull-back $f^*[(V,\sigma)]$ of the relative spin structure. (See also Subsection 3.1.)

Now we consider the case that $\tau : M \to M$ is an anti-symplectic involution and

$$L = \mathrm{Fix}\ \tau.$$

We assume L is nonempty, oriented and relatively spin. Take an ω-compatible almost complex structure J satisfying $\tau^*J = -J$. We call such J *τ-anti-invariant*. Then we find that τ induces a map

$$\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta) \longrightarrow \mathcal{M}^{\mathrm{reg}}(J;\beta)$$

which satisfies $\tau_* \circ \tau_* = \mathrm{Id}$. (See Definition 4.2 and Lemma 4.4.) Here we note that $\tau_*(\beta) = \beta$ in $\Pi(L)$ (see Remark 4.3). We pick a conjugacy class of relative spin structure $[(V,\sigma)]$ and consider the pull back $\tau^*[(V,\sigma)]$. Then we have an induced map

$$\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}.$$

The first problem we study is the question whether τ_* respects the orientation or not. The following theorem plays a fundamental role in this paper.

Theorem 1.1 (Theorem 4.5)**.** *Let L be a fixed point set of an anti-symplectic involution τ on (M,ω) and J a τ-anti-invariant almost complex structure compatible with ω. Suppose that L is oriented and carries a relative spin structure (V,σ). Then the map $\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation preserving if $\mu_L(\beta) \equiv 0 \mod 4$ and is orientation reversing if $\mu_L(\beta) \equiv 2 \mod 4$.*

Remark 1.2. If L has a *τ-relative spin structure* (see Definition 3.11), then

$$\mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} = \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$$

as oriented spaces. Corollary 4.6 is nothing but this case. If L is spin, then it is automatically τ-relatively spin (see Example 3.12). Later in Proposition 3.14 we show that there is an example of Lagrangian submanifold L which is relatively spin but not τ-relatively spin.

Including marked points, we consider the moduli space $\mathcal{M}_{k+1,m}(J;\beta)$ of J-holomorphic stable maps to (M,L) from a bordered Riemann surface $(\Sigma,\partial\Sigma)$ in class $\beta \in \Pi(L)$ of genus 0 with $(k+1)$ boundary marked points and m interior marked points. The anti-symplectic involution τ also induces a map τ_* on the moduli space with J-holomorphic maps with marked points. See Theorem 4.9. Then we have:

Theorem 1.3 (Theorem 4.9)**.** *The induced map*

$$\tau_* : \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$$

is orientation preserving if and only if $\mu_L(\beta)/2 + k + 1 + m$ is even.

When we construct the filtered A_∞ algebra $(C(L,\Lambda_{0,nov}),\mathfrak{m})$ associated to a relatively spin Lagrangian submanifold L, we use the component of $\mathcal{M}_{k+1}(J;\beta)$ consisting of the elements whose boundary marked points lie in counter-clockwise cyclic order on $\partial\Sigma$. We also involve interior marked points. For the case of $(k+1)$ boundary marked points on $\partial\Sigma$ and m interior marked points in $\operatorname{Int}\Sigma$, we denote the corresponding component by $\mathcal{M}^{\rm main}_{k+1,m}(J;\beta)$ and call it a *main component.* Moreover, we consider the moduli space $\mathcal{M}^{\rm main}_{k+1,m}(J;\beta;P_1,\dots,P_k)$ which is defined by taking a fiber product of $\mathcal{M}^{\rm main}_{k+1,m}(J;\beta)$ with smooth singular simplices $P_1,\dots,P_k$ of L. (This is nothing but the main component of (2.4) with $m=0$.) A stable conjugacy class of a relative spin structure determines orientations on these spaces as well. See Sections 2 and 3 for the definitions and a precise description of their orientations. Here we should note that τ_* above does *not* preserve the cyclic ordering of boundary marked points and so it does not preserve the main component. However, we can define the maps denoted by

$$\tau^{\rm main}_* : \mathcal{M}^{\rm main}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\rm main}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$$

and

$$\tau^{\rm main}_* : \mathcal{M}^{\rm main}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\rm main}_{k+1,m}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]} \tag{1.1}$$

respectively. See (4.7), (4.8) and (4.11) for the definitions. We put $\deg' P = \deg P - 1$ which is the shifted degree of P as a singular cochain of L (i.e., $\deg P = \dim L - \dim P$.) Then we show the following:

Theorem 1.4 (Theorem 4.11)**.** *Denote*

$$\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + m + \sum_{1\le i<j\le k} \deg' P_i \deg' P_j.$$

Then the map induced by the involution τ

$$\tau^{\rm main}_* : \mathcal{M}^{\rm main}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}^{\rm main}_{k+1,m}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]}$$

is orientation preserving if ϵ is even, and orientation reversing if ϵ is odd.

See Theorem 5.1 for a more general statement involving the fiber product with singular simplicies Q_j $(j=1,\dots,m)$ of M.

These results give rise to some non-trivial applications to Lagrangian intersection Floer theory for the case $L = \operatorname{Fix}\tau$. We briefly describe some consequences in the rest of this section.

In the book [FOOO3], using the moduli spaces $\mathcal{M}^{\rm main}_{k+1}(J;\beta;P_1,\dots,P_k)$, we construct a filtered A_∞ algebra $(C(L;\Lambda^{\mathbb{Q}}_{0,nov}),\mathfrak{m})$ with $\mathfrak{m} = \{\mathfrak{m}_k\}_{k=0,1,2,\dots}$ for any relatively spin closed Lagrangian submanifold L of (M,ω) (see Theorem 6.2) and developed the obstruction and deformation theory of Lagrangian intersection Floer cohomology. Here $\Lambda^{\mathbb{Q}}_{0,nov}$ is the universal Novikov ring over $\mathbb{Q}$ (see (6.1)). In particular, we formulate the unobstructedness to defining Floer cohomology of L as an existence of solutions of the Maurer-Cartan equation for the filtered A_∞ algebra (see Definition 6.3). We denote the set of such solutions by $\mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,nov})$. By definition, when $\mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,nov}) \neq \emptyset$, we can use any element $b \in \mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,nov})$ to deform the Floer's 'boundary' map and define a deformed Floer cohomology $HF((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,nov})$. See Subsection 6.1 for a short review of this process.

Now for the case $L = \mathrm{Fix}\ \tau$, Theorem 1.4 yields the following particular property of the filtered A_∞ algebra:

Theorem 1.5. *Let M be a compact, or tame, symplectic manifold and τ an anti-symplectic involution. If $L = \mathrm{Fix}\ \tau$ is non-empty, compact, oriented and τ-relatively spin, then the filtered A_∞ algebra $(C(L;\Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{m})$ can be chosen so that*

$$\mathfrak{m}_{k,\beta}(P_1, \dots, P_k) = (-1)^{\epsilon_1}\mathfrak{m}_{k,\tau_*\beta}(P_k, \dots, P_1) \tag{1.2}$$

where

$$\epsilon_1 = \frac{\mu_L(\beta)}{2} + k + 1 + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j.$$

Using the results from [FOOO3], we derive that Theorem 1.5 implies unobstructedness of $L = \mathrm{Fix}\ \tau$ in the following cases:

Corollary 1.6. *Let τ and $L = \mathrm{Fix}\ \tau$ be as in Theorem 1.5. In addition, we assume either (i) $c_1(TM)|_{\pi_2(M)} \equiv 0 \mod 4$ or (ii) $c_1(TM)|_{\pi_2(M)} \equiv 0 \mod 2$ and $i_* : \pi_1(L) \to \pi_1(M)$ is injective. (Here i_* is the natural map induced by inclusion $i : L \to M$.) Then L is unobstructed over $\Lambda_{0,nov}^{\mathbb{Q}}$ (i.e., $\mathcal{M}(L;\Lambda_{0,nov}^{\mathbb{Q}}) \neq \emptyset$) and so $HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}})$ is defined for any $b \in \mathcal{M}(L;\Lambda_{0,nov}^{\mathbb{Q}})$. Moreover, we may choose $b \in \mathcal{M}(L;\Lambda_{0,nov}^{\mathbb{Q}})$ so that the map*

$$\begin{aligned}(-1)^{k(\ell+1)}(\mathfrak{m}_2)_* : HF^k((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}}) \otimes HF^\ell((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}}) \\ \longrightarrow HF^{k+\ell}((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}})\end{aligned}$$

induces a graded commutative product.

Remark 1.7. By symmetrizing the filtered A_∞ structure $\mathfrak{m}_k$ of $(C(L,\Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{m})$, we obtain a filtered L_∞ algebra $(C(L,\Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{l}) = (C(L,\Lambda_{0,nov}^{\mathbb{Q}}), \{\mathfrak{l}_k\}_{k=0,1,2,\dots})$. See Section A3 [FOOO3] for the definitions of the symmetrization and of the filtered L_∞ structure. In the situation of Corollary 1.6, the same proof shows that we have $\mathfrak{l}_k = \overline{\mathfrak{l}}_k \otimes \Lambda_{0,nov}^{\mathbb{Q}}$ if k is even. Here $\overline{\mathfrak{l}}_k$ is the (unfiltered) L_∞ structure obtained by the reduction of the coefficient of $(C(L,\Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{l})$ to $\mathbb{Q}$. Note that *over* $\mathbb{R}$ we may choose $\overline{\mathfrak{l}}_k = 0$ for $k \ge 3$ by Theorem X in Chapter 1 [FOOO3]. On the other hand, Theorem A3.19 [FOOO3] shows that $\overline{\mathfrak{l}}_k = 0$ for $H(L;\Lambda_{0,nov}^{\mathbb{Q}})$.

We note that we do not assert that Floer cohomology $HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}})$ is isomorphic to $H^*(L;\mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$. (Namely, we do not assert $\mathfrak{m}_1 = \overline{\mathfrak{m}}_1 \otimes \Lambda_{0,nov}^{\mathbb{Q}}$.) Indeed, we show in Subsection 6.3 that for the case $L = \mathbb{R}P^{2n+1}$ in $\mathbb{C}P^{2n+1}$ the Floer cohomology group is *not* isomorphic to the classical cohomology group. (See Theorem 6.26.)

Moreover, if we assume $c_1(TM)|_{\pi_2(M)} = 0$ in addition, we can show the following non-vanishing theorem of Floer cohomology:

Corollary 1.8. *Let τ, $L = \mathrm{Fix}\ \tau$ be as in Theorem 1.5. Assume $c_1(TM)|_{\pi_2(M)} = 0$. Then L is unobstructed over $\Lambda_{0,nov}^{\mathbb{Q}}$ and*

$$HF((L,b),(L,b);\Lambda_{nov}^{\mathbb{Q}}) \neq 0$$

for any $b \in \mathcal{M}(L;\Lambda_{0,nov}^{\mathbb{Q}})$. In particular, we have

$$\psi(L) \cap L \neq \emptyset$$

for any Hamiltonian diffeomorphism $\psi : M \to M$.

Theorem 1.5 and Corollaries 1.6, 1.8 can be applied to the real point set L of any Calabi-Yau manifold (defined over $\mathbb{R}$) if it is oriented and τ-relative spin.

Another application of Theorem 1.5 and Corollary 1.6 is a ring isomorphism between quantum cohomology and Lagrangian Floer cohomology for the case of the diagonal of square of a symplectic manifold. Let (N, ω_N) be a closed symplectic manifold. We consider the product

$$(M, \omega_M) = (N \times N, -p_1^*\omega_N + p_2^*\omega_N),$$

where p_i is the projection to the i-th factor. The involution $\tau : M \to M$ defined by $\tau(x, y) = (y, x)$ is anti-symplectic and its fixed point set L is the diagonal

$$\Delta_N = \{(x, x) \mid x \in N\} \cong N.$$

As we will see in the proof of Theorem 1.9, the diagonal set is always unobstructed. Moreover, we note that the natural map $i_* : H_*(\Delta_N, \mathbb{Q}) \to H_*(N \times N; \mathbb{Q})$ is injective and so the spectral sequence constructed in Chapter 6 [FOOO3] collapses at E_2-term by Theorem D (D.3) [FOOO3], which in turn induces the natural isomorphism $H(N; \mathbb{Q}) \otimes \Lambda_{0,nov} \cong HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}})$ for any $b \in \mathcal{M}(L; \Lambda_{0,nov}^{\mathbb{Q}})$. We prove in the proof of Theorem 1.9 in Subsection 6.2 that $\mathfrak{m}_2$ also derives a graded commutative product

$$\cup_Q : HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \otimes HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \to HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}).$$

In fact, we can prove that the following stronger statement.

Theorem 1.9. *Let* (N, ω_N) *be a closed symplectic manifold.*

(1) *The diagonal set of* $(N \times N, -p_1^*\omega_N + p_2^*\omega_N)$ *is unobstructed over* $\Lambda_{0,nov}^{\mathbb{Q}}$.
(2) *The product* $\cup_Q$ *coincides with the quantum cup product on* (N, ω_N) *under the natural isomorphism* $HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \cong H(N; \mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$.

If we use Corollary 3.8.43 [FOOO3], we can easily find that the diagonal set is *weakly unobstructed* in the sense of Definition 3.6.29 [FOOO3]. See also Remark 6.27. We also note that for the case of diagonals, $\mathfrak{m}_k$ ($k \geq 3$) define a quantum (higher) Massey product. It was discussed formally in [F1]. We have made it rigorous here:

Corollary 1.10. *For any closed symplectic manifold* (N, ω_N)*, there exists a filtered* A_∞ *structure* $\mathfrak{m}_k$ *on* $H(N; \Lambda_{0,nov}^{\mathbb{Q}}) = H(N; \mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$ *such that*

(1) $\mathfrak{m}_0 = \mathfrak{m}_1 = 0$;
(2) $\cup_Q$ *defined by (6.10) using* $\mathfrak{m}_2$ *coincides with the quantum cup product;*
(3) *the* $\mathbb{R}$*-reduction* $(H(N; \mathbb{Q}), \overline{\mathfrak{m}}) \otimes_{\mathbb{Q}} \mathbb{R}$ *of the filtered* A_∞ *algebra is homotopy equivalent to the de Rham complex of* N *as an* A_∞ *algebra, where* $(H(N; \mathbb{Q}), \overline{\mathfrak{m}})$ *is the reduction of the coefficient of* $(H(N; \Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{m})$ *to* $\mathbb{Q}$.

The paper is organized as follows: In Section 2, we briefly recall some basic material on the moduli space of stable maps from a bordered Riemann surface of genus 0. In Section 3, we also recall from [FOOO3] the notion of relative spin structure, its stable conjugacy class and the orientation of the moduli space of pseudo-holomorphic discs. We describe how the stable conjugacy class of relative spin structure determines an orientation on the moduli space. We introduce here the notion of τ-relative spin structure for an anti-symplectic involution $\tau : M \to M$,

and also give some examples which are relatively spin but not τ-relatively spin Lagrangian submanifolds. In Section 4, we define the map τ_* on the moduli space induced by τ and study how the induced map τ_* on various moduli spaces changes or preserves the orientations. Assuming Theorem 1.1 holds, we prove Theorem 1.3 in this section. The fundamental theorems Theorem 1.1 and Theorem 1.4 are proved in Section 5. Section 6 is devoted to various applications of the results obtained above to Lagrangian Floer cohomology. After a short review of general story of Lagrangian intersection Floer theory laid out in [FOOO3], we prove Theorem 1.5, Corollary 1.6, Corollary 1.8 Theorem 1.9 and Corollary 1.10 in Subsection 6.2. In Subsection 6.3, we calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda^{\mathbb{Z}}_{0,nov}$ coefficients by studying orientations in detail. The calculation shows that the Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda^{\mathbb{Z}}_{0,nov}$ is not isomorphic to the usual cohomology. This result contrasts with Oh's earlier calculation [Oh1] of the Floer cohomology of real projective spaces over $\mathbb{Z}_2$ coefficients, where the Floer cohomology is isomorphic to the usual cohomology over $\mathbb{Z}_2$. In Appendix, we briefly recall form [FOOO3] the definition of orientation on the space with Kuranishi structure and the notion of group action on the space with Kuranishi structure.

Originally, the content of this paper appeared as a part of Chapter 8 in the preprint version [FOOO2] of the book [FOOO3] and was intended to be published in a part of the book. However, due to the publisher's page restriction on the AMS/IP Advanced Math Series, we took out two chapters, Chapter 8 and Chapter 10 from the preprint [FOOO2] and published the book without those two chapters. The content of this paper is a rearrangement and combination of the parts extracted from Chapter 8 (Floer theory of Lagrangian submanifolds over $\mathbb{Z}$) and Chapter 9 (Orientation) in the preprint [FOOO2]. We also note that this is a part of the paper cited as [FOOO09I] in the book [FOOO3].

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2. Preliminaries

In this section, we prepare some basic notations we use in this paper. We refer Section 2.1 and Section A1 in [FOOO3] for more detailed explanation of moduli spaces and the notion of Kuranishi structure, respectively. Let L be an oriented compact Lagrangian submanifold of (M,ω). Take an ω-compatible almost complex structure J on M. We recall Definition 2.4.17 in [FOOO3] where we introduce the relation $\sim$ in $\pi_2(M,L)$: We define $\beta \sim \beta'$ in $\pi_2(M,L)$ if and only if $\omega(\beta)=\omega(\beta')$ and $\mu_L(\beta)=\mu_L(\beta')$. We denote the quotient group by

$$\Pi(L)=\pi_2(M,L)/\sim . \tag{2.1}$$

This is an abelian group. Let $\beta \in \Pi(L)$. *A stable map from a bordered Riemann surface of genus zero with $(k+1)$ boundary marked points and m interior marked points* is a pair $((\Sigma,\vec z,\vec z^+),w)=((\Sigma,z_0,\dots,z_k,z_1^+,\dots,z_m^+),w)$ such that $(\Sigma,\vec z,\vec z^+)$ is a bordered semi-stable curve of genus zero with $(k+1)$ boundary marked points

and m interior marked points and $w : (\Sigma, \partial\Sigma) \to (M, L)$ is a J-holomorphic map such that its automorphism group, i.e., the set of biholomorphic maps $\psi : \Sigma \to \Sigma$ satisfying $\psi(z_i) = z_i, \psi(\vec{z}_i^{\,+}) = \vec{z}_i^{\,+}$ and $w \circ \psi = w$ is finite. We say that $((\Sigma, \vec{z}, \vec{z}^{\,+}), w)$ is isomorphic to $((\Sigma', \vec{z}', \vec{z}^{\,+\prime}), w')$, if there exists a holomorphic map $\psi : \Sigma \to \Sigma'$ satisfying $\psi(z_i) = z_i', \psi(\vec{z}_i^{\,+}) = \vec{z}_i^{\,+\prime}$ and $w' \circ \psi = w$. We denote by $\mathcal{M}_{k+1,m}(J;\beta)$ the set of the isomorphism classes of stable maps in class $\beta \in \Pi(L)$ from a bordered Riemann surface of genus zero with $(k+1)$ boundary marked points and m interior marked points. When the domain curve Σ is a smooth 2-disc D^2, we denote the corresponding subset by $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$. We note that $\mathcal{M}_{k+1,m}(J;\beta)$ is a compactification of $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$. The virtual real dimension is

$$\dim_{\mathbb{R}} \mathcal{M}_{k+1,m}(J;\beta) = n + \mu_L(\beta) + k + 1 + 2m - 3,$$

where $n = \dim L$ and $\mu_L(\beta)$ is the Maslov index which is an even integer for an oriented Lagrangian submanifold L. When we do not consider interior marked points, we denote them by $\mathcal{M}_{k+1}(J;\beta)$, $\mathcal{M}^{\mathrm{reg}}_{k+1}(J;\beta)$, and when we do not consider any marked points, we simply denote them by $\mathcal{M}(J;\beta)$, $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ respectively. Furthermore, we define a component $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ of $\mathcal{M}_{k+1,m}(J;\beta)$ by

$$\begin{aligned}\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta) = \{&((\Sigma, \vec{z}, \vec{z}^{\,+}), w) \in \mathcal{M}_{k+1,m}(J;\beta) \mid \\ &(z_0, z_1, \dots, z_k) \text{ is in counter-clockwise cyclic order on } \partial\Sigma\},\end{aligned}$$

which we call the *main component*. We define $\mathcal{M}^{\mathrm{main,reg}}_{k+1,m}(J;\beta)$, $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta)$ and $\mathcal{M}^{\mathrm{main,reg}}_{k+1}(J;\beta)$ in a similar manner.

We have a Kuranishi structure on $\mathcal{M}_{k+1,m}(J;\beta)$ so that the evaluation maps

$$\begin{aligned} ev_i &: \mathcal{M}_{k+1,m}(J;\beta) \longrightarrow L, \quad i = 0, 1, \dots, k \\ ev_j^+ &: \mathcal{M}_{k+1,m}(J;\beta) \longrightarrow M, \quad j = 1, \dots, m \end{aligned} \tag{2.2}$$

defined by $ev_i((\Sigma, \vec{z}, \vec{z}^{\,+}), w)) = w(z_i)$ and $ev_j^+((\Sigma, \vec{z}, \vec{z}^{\,+}), w)) = w(z_j^+)$ are weakly submersive. (See Section 5 [FO] and Section A1.1 [FOOO3] for the definitions of Kuranishi structure and weakly submersive map.) Then for given smooth singular simplicies $(f_i : P_i \to L)$ of L and $(g_j : Q_j \to M)$ of M, we can define the fiber product in the sense of Kuranishi structure:

$$\begin{aligned}&\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P}) \\ :=&\mathcal{M}_{k+1,m}(J;\beta)_{(ev_1^+,\dots,ev_m^+,ev_1,\dots,ev_k)} \times_{g_1\times\dots\times f_k} \left(\prod_{j=1}^m Q_j \times \prod_{i=1}^k P_i\right).\end{aligned} \tag{2.3}$$

See Section A1.2 [FOOO3] for the definition of fiber product of Kuranishi structures. We define $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$ in a similar way. When we do not consider the interior marked points, we denote the corresponding moduli spaces by $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;\vec{P})$, respectively. Namely,

$$\mathcal{M}_{k+1}(J;\beta;\vec{P}) := \mathcal{M}_{k+1}(J;\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\dots\times f_k} \left(\prod_{i=1}^k P_i\right). \tag{2.4}$$

In Subsection 3.3, we describe the orientations on these spaces precisely.

3. τ-relative spin structure and orientation

3.1. **Review of relative spin structure and orientation.** It is known that the moduli space of pseudo-holomorphic discs with Lagrangian boundary condition is not always orientable. To discuss orientability and orientation of the moduli space, we first recall the notion of relative spin structure and its stable conjugacy class introduced in [FOOO3] and also briefly review how the stable conjugacy class of relative spin structure determines an orientation of the moduli space of pseudo-holomorphic discs with Lagrangian boundary condition. See Section 8.1 [FOOO3] for more details. See also V.de Silva's work [Si].

Definition 3.1. An oriented Lagrangian submanifold L of M is called *relatively spin* if there exists a class $st \in H^2(M;\mathbb{Z}_2)$ such that $st|_L = w_2(TL)$.

A pair of Lagrangian submanifolds $(L^{(1)}, L^{(0)})$ is called *relatively spin*, if there exists a class $st \in H^2(M;\mathbb{Z}_2)$ satisfying $st|_{L^{(i)}} = w_2(TL^{(i)})$ $(i = 0, 1)$ simultaneously.

Remark 3.2. Using the pin structure, J. Solomon [So] generalized our results about orientation problem studied in [FOOO3] to the case of non-orientable Lagrangian submanifolds.

Let L be a relatively spin Lagrangian submanifold of M. We fix a triangulation of M such that L is a subcomplex. A standard obstruction theory yields that we can take an oriented real vector bundle V over the 3-skeleton $M_{[3]}$ of M which satisfies $w_2(V) = st$. Then $w_2(TL|_{L_{[2]}} \oplus V|_{L_{[2]}}) = 0$ and so $TL \oplus V$ carries a spin structure on the 2-skeleton $L_{[2]}$ of L.

Definition 3.3. The choice of an orientation on L, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$, an oriented real bector bundle V over the 3-skeleton $M_{[3]}$ satisfying $w_2(V) = st$ and a spin structure σ on $(TL \oplus V)|_{L_{[2]}}$ is called a *relative spin structure* on $L \subset M$.

A *relative spin structure* on the pair $(L^{(1)}, L^{(0)})$ is the choice of orientations on $L^{(i)}$, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$, an oriented real vector bundle V over the 3 skeleton $M_{[3]}$ satisfying $w_2(V) = st$ and spin structures on $(TL^{(i)} \oplus V)|_{L^{(i)}_{[2]}}$ $(i = 0, 1)$.

In this paper we fix an orientation on L. If L is spin, we have an associated relative spin structure for each spin structure on L as follows: Take $st = 0$ and V is trivial. Then the spin structure on L naturally induces the spin structure on $TL \oplus V$.

Definition 3.3 depends on the choices of V and the triangulation of M. We introduce an equivalence relation called *stable conjugacy* on the set of relative spin structures so that the stable conjugacy class is independent of such choices.

Definition 3.4. We say that two relative spin structures (st_i, V_i, σ_i) $(i = 1, 2)$ on L are *stably conjugate*, if there exist integers k_i and an orientation preserving bundle isomorphism $\varphi : V_1 \oplus \mathbb{R}^{k_1} \to V_2 \oplus \mathbb{R}^{k_2}$ such that by $1 \oplus \varphi|_{L_{[2]}} : (TL \oplus V_1)_{L_{[2]}} \oplus \mathbb{R}^{k_1} \to (TL \oplus V_2)_{L_{[2]}} \oplus \mathbb{R}^{k_2}$, the spin structure $\sigma_1 \oplus 1$ induces the spin structure $\sigma_2 \oplus 1$.

Here $\mathbb{R}^{k_i}$ denote trivial vector bundles of rank k_i $(i = 1, 2)$. We note that in Definition 3.4, we still fix a triangulation of M such that L is a subcomplex. However, by Proposition 8.1.6 in [FOOO3], we find that the stable conjugacy class

of relative spin structure is actually independent of the choice of a triangulation of M as follows: We denote by $\mathrm{Spin}(M, L; \mathfrak{T})$ the set of all the stable conjugacy classes of relative spin structures on $L \subset M$.

Proposition 3.5 (Proposition 8.1.6 in [FOOO3])**.** (1) *There is a simply transitive action of* $H^2(M, L; \mathbb{Z}_2)$ *on* $\mathrm{Spin}(M, L; \mathfrak{T})$.

(2) *For two triangulations* $\mathfrak{T}$ *and* $\mathfrak{T}'$ *of* M *such that* L *is a subcomplex, there exists a canonical isomorphism* $\mathrm{Spin}(M, L; \mathfrak{T}) \cong \mathrm{Spin}(M, L; \mathfrak{T}')$ *compatible with the above action.*

In particular, if a spin structure of L is given, there is a canonical isomorphism $\mathrm{Spin}(M, L; \mathfrak{T}) \cong H^2(M, L; \mathbb{Z}_2)$. Thus, hereafter, we denote by $\mathrm{Spin}(M, L)$ the set of the stable conjugacy classes of relative spin structures on L without specifying any triangulation of M.

Since the class st is determined by V, we simply write the stable conjugacy class of relative spin structure as $[(V, \sigma)]$ where σ is a spin structure on $(TL \oplus V)|_{L_{[2]}}$.

The following theorem is proved in Section 8.1 [FOOO3]. We denote by $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$ the set of all J-holomorphic maps from $(D^2, \partial D^2)$ to (M, L) representing a class β. We note that $\mathcal{M}^{\mathrm{reg}}(J; \beta) = \widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)/PSL(2, \mathbb{R})$.

Theorem 3.6. *If L is a relatively spin Lagrangian submanifold,* $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$ *is orientable. Furthermore, the choice of stable conjugacy class of relative spin structure on L determines an orientation on* $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$ *canonically for all* $\beta \in \pi_2(M, L)$.

Remark 3.7. (1) Following Convention 8.2.1 in [FOOO3], we have an induced orientation on the quotient space. Thus Theorem 3.6 holds for the quotient space $\mathcal{M}^{\mathrm{reg}}(J; \beta) = \widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)/PSL(2, \mathbb{R})$ as well. Here we use the orientation of $PSL(2, \mathbb{R})$ as in Convention 8.3.1 in [FOOO3].

(2) Since $\mathcal{M}^{\mathrm{reg}}(J; \beta)$ is the top dimensional stratum of $\mathcal{M}(J; \beta)$, the orientation on $\mathcal{M}^{\mathrm{reg}}(J; \beta)$ determines one on $\mathcal{M}(J; \beta)$. In this sense, it is enough to consider $\mathcal{M}^{\mathrm{reg}}(J; \beta)$ when we discuss orientation on $\mathcal{M}(J; \beta)$. The same remark applies to other moduli spaces including marked points and fiber products with singular simplices.

We recall from Section 8.1 [FOOO3] how each stable conjugacy class of relative spin structure determines an orientation on the moduli space of holomorphic discs. Once we know orientability of $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$, it suffices to give an orientation on the determinant of tangent space at a point $w \in \widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$ for each stable conjugacy class of relative spin structure. We consider the linearized operator of the pseudo-holomorphic curve equation

$$D_w \overline{\partial} : W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to L^p(D^2; w^*TM \otimes \Lambda^{0,1}_{D^2}). \tag{3.1}$$

Here $\ell = w|_{\partial D^2}$ and $p > 2$. Since it has the same symbol as the Dolbeault operator

$$\overline{\partial}_{(w^*TM, \ell^*TL)} : W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to L^p(D^2; w^*TM \otimes \Lambda^{0,1}_{D^2}),$$

we may consider the determinant of the index of this Dolbeault operator $\overline{\partial}_{(w^*TM, \ell^*TL)}$ instead. We can deform $w : (D^2, \partial D^2) \to (M, L)$ to $w_0 : (D^2, \partial D^2) \to (M_{[2]}, L_{[1]})$ by the simplicial approximation theorem. We put $\ell_0 = w_0|_{\partial D^2}$.

Now pick $[(V, \sigma)] \in \mathrm{Spin}(M, L)$. Then it determines the stable homotopy class of trivialization of $\ell_0^*(TL \oplus V)$. The existence of the oriented bundle V on $M_{[3]}$ induces a unique homotopy class of trivialization of $\ell_0^* V$. Thus, we have a unique

homotopy class of trivialization of $\ell_0^* TL$. Using this trivialization and Proposition 3.8 below (applied to the pair of (E, λ) with $E = w_0^* TM, \lambda = \ell_0^* TL$), we can assign an orientation on the determinant of the index

$$\det \mathrm{Index}\ \overline{\partial}_{(w_0^* TM, \ell_0^* TL)}) := \det(\mathrm{Coker}\ \overline{\partial}_{(w_0^* TM, \ell_0^* TL)})^* \otimes \det \mathrm{Ker}\ \overline{\partial}_{(w_0^* TM, \ell_0^* TL)}.$$

This process is invariant under stably conjugate relation of relative spin structures. Therefore we obtain an orientation on $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ and so on $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ for each stable conjugacy class of relative spin structure $[(V, \sigma)]$.

Proposition 3.8 (Proposition 8.1.4 [FOOO3])**.** *Let E be a complex vector bundle over D^2 and λ a maximally totally real bundle over ∂D^2 with an isomorphism*

$$E|_{\partial D^2} \cong \lambda \otimes \mathbb{C}.$$

Suppose that λ is trivial. Then each trivialization on λ canonically induces an orientation on $\mathrm{Index}\,\overline{\partial}_{(E,\lambda)}$. *Here $\overline{\partial}_{(E,\lambda)}$ is the Dolbeault operator on $(D^2, \partial D^2)$ with coefficient (E, λ):*

$$\overline{\partial}_{(E,\lambda)} : W^{1,p}(D^2, \partial D^2; E, \lambda) \to L^p(D^2; E \otimes \Lambda^{0,1}_{D^2}).$$

Remark 3.9. In order to explain some part of the proof of Theorem 1.1 given in Section 5 in a self-contained way, we briefly recall the outline of the proof of Proposition 3.8. See Subsection 8.1.1 [FOOO3] for more detail. For $0 < \epsilon < 1$, we put $A(\epsilon) = \{z \in D^2 \mid 1 - \epsilon \le |z| \le 1\}$ and $C_{1-\epsilon} = \{z \in D^2 \mid |z| = 1 - \epsilon\}$. By pinching the circle C to a point, we have a union of a 2-disc D^2 and a 2-sphere $\mathbb{C}P^1$ with the center $O \in D^2$ identified with a point $p \in \mathbb{C}P^1$. The resulting space $\Sigma = D^2 \cup \mathbb{C}P^1$ has naturally a structure of a nodal curve where $O = p$ is the nodal point. Under the situation of Proposition 3.8, the trivial bundle $\lambda \to \partial D^2$ trivially extends to $A(\epsilon)$ and the complexification of each trivialization of $\lambda \to \partial D^2$ gives a trivialization on $E|_{A(\epsilon)} \to A(\epsilon)$. Thus the bundle $E \to D^2$ descends to a bundle over the nodal curve Σ together with a maximally totally real bundle over $\partial\Sigma = \partial D^2$. We denote them by $E' \to \Sigma$ and $\lambda' \to \partial\Sigma$ respectively. We also denote by $W^{1,p}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$ the space of $W^{1,p}$-sections of $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$ and by $W^{1,p}(D^2; E'|_{D^2}, \lambda')$ the space of $W^{1,p}$-sections ξ_{D^2} of $E'|_{D^2} \to D^2$ satisfying $\xi_{D^2}(z) \in \lambda'_z$, $z \in \partial D^2 = \partial\Sigma$. We consider a map denoted by diff:

$$\begin{aligned} \mathrm{diff} : W^{1,p}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \oplus W^{1,p}(D^2, \partial D^2; E'|_{D^2}, \lambda') &\to \mathbb{C}^n; \\ (\xi_{\mathbb{C}P^1}, \xi_{D^2}) &\mapsto \xi_{\mathbb{C}P^1}(p) - \xi_{D^2}(O). \end{aligned}$$

We put $W^{1,p}(E', \lambda') := \mathrm{diff}^{-1}(0)$ and consider the index of operator

$$\overline{\partial}_{(E',\lambda')} \ : \ W^{1,p}(E', \lambda') \to L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1}) \oplus L^p(D^2, \partial D^2; E'|_{D^2} \otimes \Lambda^{0,1}_{D^2}).$$

Then the orientation problem for $\mathrm{Index}\,\overline{\partial}_{(E,\lambda)}$ on $(D^2, \partial D^2)$ is translated into the problem for $\mathrm{Index}\,\overline{\partial}_{(E',\lambda')}$ on $(\Sigma, \partial\Sigma)$. Firstly, we note that the operator

$$\overline{\partial}_{(E'|_{D^2}, \lambda'|_{\partial D^2})} \ : \ W^{1,p}(D^2, \partial D^2; E'|_{D^2}, \lambda') \to L^p(D^2; E'|_{D^2} \otimes \Lambda^{0,1}_{D^2})$$

is surjective. Each trivialization of $\lambda \to \partial D^2$ gives an identification:

$$\mathrm{Ker}\,\overline{\partial}_{(E'|_{D^2}, \lambda'|_{\partial D^2})} \cong \mathrm{Ker}\,\overline{\partial}_{(D^2 \times \mathbb{C}^n, \partial D^2 \times \mathbb{R}^n)} \cong \mathbb{R}^n, \tag{3.2}$$

where $\mathrm{Ker}\,\overline{\partial}_{(D^2 \times \mathbb{C}^n, \partial D^2 \times \mathbb{R}^n)}$ is the space of solutions $\xi : D^2 \to \mathbb{C}^n$ of the Cauchy-Riemann equation with boundary condition:

$$\overline{\partial}\xi = 0, \quad \xi(z) \in \lambda'_z \equiv \mathbb{R}^n, \quad z \in \partial D^2.$$

Thus the solution must be a real constant vector. This implies that we have a canonical isomorphism in (3.2). Then the argument in Subsection 8.1.1 [FOOO3] shows that the orientation problem can be reduced to the orientation on $\operatorname{Ker}\overline{\partial}_{(E'|_{D^2},\lambda'|_{\partial D^2})}$ and $\operatorname{Index}\overline{\partial}_{E'|_{\mathbb{C}P^1}}$. The latter one has a complex orientation. By taking a finite dimensional complex vector space $W \subset L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1})$ such that

$$L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1}) = \operatorname{Image}\overline{\partial}_{E'|_{\mathbb{C}P^1}} + W,$$

a standard argument (see the paragraphs after Remark 8.1.3 in [FOOO3], for example) shows that the orientation problem on $\operatorname{Index}\overline{\partial}_{E'|_{\mathbb{C}P^1}}$ is further reduced to one on $\operatorname{Ker}\overline{\partial}_{E'|_{\mathbb{C}P^1}}$ which is the space of holomorphic sections of $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$, denoted by $\operatorname{Hol}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$.

We next describe how the orientation behaves under the change of stable conjugacy classes of relative spin structures. Proposition 3.5 shows that the difference of relative spin structures is measured by an element $\mathfrak{x}$ in $H^2(M, L; \mathbb{Z}_2)$. We denote the simply transitive action of $H^2(M, L; \mathbb{Z}_2)$ on $\operatorname{Spin}(M, L)$ by

$$(\mathfrak{x}, [(V, \sigma)]) \mapsto \mathfrak{x} \cdot [(V, \sigma)].$$

When we change the relative spin structure by $\mathfrak{x} \in H^2(M, L; \mathbb{Z}_2)$, then we find that the orientation on the index of the operator $D_w\overline{\partial}$ in (3.1) changes by $(-1)^{\mathfrak{x}[w]}$. The following result is proved in Proposition 8.1.16 in [FOOO3] and also obtained by Cho [C]. This proposition is used in Subsections 6.2 and 6.3.

Proposition 3.10. *The identity map*

$$\mathcal{M}(J; \beta)^{[(V,\sigma)]} \longrightarrow \mathcal{M}(J; \beta)^{\mathfrak{x}\cdot[(V,\sigma)]}$$

is orientation preserving if and only if $\mathfrak{x}[\beta] = 0$.

For a diffeomorphism $\psi : (M, L) \to (M', L')$ satisfying $\psi(L) = L'$, we define the pull-back map

$$\psi^* \ : \ \operatorname{Spin}(M', L') \longrightarrow \operatorname{Spin}(M, L) \tag{3.3}$$

by $\psi^*[(V', \sigma')] = [(\psi^* V', \psi^* \sigma')]$. That is, we take a triangulation on M' such that L' is its subcomplex and $\psi : (M, L) \to (M', L')$ is a simplicial map. Then $\psi^* V'$ is a real vector bundle over $M_{[3]}$ and σ' induces a spin structure on $(TL \oplus \psi^* V')|_{L_{[2]}}$. Then it is easy to see that

$$\psi_* \ : \ \mathcal{M}(\beta; M, L; J)^{\psi^*[(V',\sigma')]} \longrightarrow \mathcal{M}(\beta; M', L'; \psi_* J)^{[(V',\sigma')]}$$

is orientation preserving.

3.2. **τ-relative spin structure and an example.** After these general results are prepared in the previous subsection, we focus ourselves on the case $L = \operatorname{Fix} \tau$, the fixed point set of an anti-symplectic involution τ of M. We define the notion of τ-relative spin structure and discuss its relationship with the orientation of the moduli space. Note that for τ such that $L = \operatorname{Fix} \tau$ is relative spin, τ induces an involution τ^* on the set of relative spin structures (V, σ) by pull-back (3.3).

Definition 3.11. A *τ-relative spin structure* on L is a relative spin structure (V, σ) on L such that $\tau^*(V, \sigma)$ is stably conjugate to (V, σ), i.e., $\tau^*[(V, \sigma)] = [(V, \sigma)]$ in $\operatorname{Spin}(M, L)$. We say that L is *τ-relatively spin* if it carries a τ-relative spin structure i.e., if the involution $\tau^* : \operatorname{Spin}(M, L) \to \operatorname{Spin}(M, L)$ has a fixed point.

Example 3.12. If L is spin, then it is τ-relatively spin: Obviously τ preserves spin structure of L since it is identity on L. And we may take V needed in the definition of relative spin structure to be the trivial vector bundle.

Remark 3.13. We like to emphasize that a relative spin structure (V, σ, st) satisfying $\tau^* st = st$ is *not necessarily* a τ-relative spin structure in the sense of Definition 3.11. See Proposition 3.14 below.

Now we give an example of $L = \mathrm{Fix}\ \tau$ that is relatively spin but *not* τ-relatively spin.

Consider $M = \mathbb{C}P^n$ with standard symplectic and complex structures and $L = \mathbb{R}P^n \subset \mathbb{C}P^n$, the real points set. The real projective space $\mathbb{R}P^n$ is oriented if and only if n is odd. We take the tautological real line bundle ξ on $\mathbb{R}P^n$ such that the 1-st Stiefel-Whitney class $w_1(\xi) := x$ is a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$. Then we have

$$T\mathbb{R}P^n \oplus \mathbb{R} \cong \xi^{\oplus(n+1)}$$

and so the total Stiefel-Whitney class is given by

$$w(T\mathbb{R}P^n) = (1+x)^{n+1}.$$

Therefore we have

$$w_2(\mathbb{R}P^{2n+1}) = \begin{cases} x^2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.4}$$

From this, it follows that $\mathbb{R}P^{4n+3}$ $(n \geq 0)$ and $\mathbb{R}P^1$ are spin and hence are τ-relatively spin by Example 3.12. On the other hand we prove:

Proposition 3.14. *The real projective space $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$ $(n \geq 1)$ is relatively spin but not τ-relatively spin.*

Proof. The homomorphism

$$H^2(\mathbb{C}P^{4n+1}; \mathbb{Z}_2) \longrightarrow H^2(\mathbb{R}P^{4n+1}; \mathbb{Z}_2)$$

is an isomorphism. We can construct an isomorphism explicitly as follows: Let η be the tautological complex line bundle on $\mathbb{C}P^{4n+1}$ such that

$$c_1(\eta) = y \in H^2(\mathbb{C}P^{4n+1}; \mathbb{Z})$$

is a generator. We can easily see that

$$\eta|_{\mathbb{R}P^{4n+1}} = \xi \oplus \xi$$

where ξ is the real line bundle over $\mathbb{R}P^{4n+1}$ chosen as above. Since $c_1(\eta)$ is the Euler class which reduces to the second Stiefel-Whitney class under $\mathbb{Z}_2$-reduction, $y \mapsto x^2$ under the above isomorphism. But (3.4) shows that $x^2 = w_2(\mathbb{R}P^{4n+1})$. This proves that $\mathbb{R}P^{4n+1}$ is relatively spin: For st, we take $st = y$.

Now we examine the relative spin structures of $\mathbb{R}P^{4n+1}$. It is easy to check that $H^2(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and so there are 2 inequivalent relative spin structures by Proposition 3.5. Let $st = y$ and take

$$V = \eta^{\oplus 2n+1} \oplus \mathbb{R}$$

for the vector bundle V noting $w_2(V) \equiv c_1(V) = (2n+1)y = y = st \mod 2$.

Next we have the isomorphism

$$\tilde{\sigma} : T\mathbb{R}P^{4n+1} \oplus \mathbb{R}^2 \cong \xi^{\oplus(4n+2)} \oplus \mathbb{R} \cong \eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}$$

and so it induces a trivialization of

$$\begin{aligned}(T\mathbb{R}P^{4n+1} \oplus V) \oplus \mathbb{R}^2 &\cong T\mathbb{R}P^{4n+1} \oplus \mathbb{R}^2 \oplus V \\ &\cong (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}) \oplus (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}).\end{aligned}$$

We note that on a 2 dimensional CW complex, any stable isomorphism between two oriented real vector bundles V_1, V_2 induces a stable trivialization of $V_1 \oplus V_2$. In particular, $(\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}) \oplus (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R})$ has a canonical stable trivialization on the 2-skeleton of $\mathbb{R}P^{4n+1}$, which in turn provides a spin structure on $T\mathbb{R}P^{4n+1} \oplus V$ denoted by σ. This provides a relative spin structure on $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$.

Next we study the question on the τ-relatively spin property. By the definition of the tautological line bundle η on $\mathbb{C}P^{4n+1}$, the involution τ lifts to an anti-complex linear isomorphism of η which we denote

$$c \,:\, \eta \longrightarrow \eta.$$

Then

$$c^{\oplus(2n+1)} \oplus (-1) \,:\, \eta^{\oplus(2n+1)} \oplus \mathbb{R} \longrightarrow \eta^{\oplus(2n+1)} \oplus \mathbb{R}$$

is an isomorphism which covers τ. Therefore we may identify

$$\tau^* V = V = \eta^{\oplus(2n+1)} \oplus \mathbb{R}$$

on $\mathbb{R}P^{4n+1}$ and also

$$\tau^* = c^{\oplus(2n+1)} \oplus (-1).$$

Then we have

$$\tau^*(V, \sigma, st) = (V, \sigma', st)$$

where the spin structure σ' corresponds to the isomorphism

$$(c^{\oplus(2n+1)} \oplus (-1)) \circ \tilde{\sigma}.$$

Therefore to complete the proof of Proposition 3.14 it suffices to show that the restriction of $c^{\oplus(2n+1)} \oplus (-1)$ to $(\mathbb{R}P^{4n+1})_{[2]}$ is not stably homotopic to the identity map as a bundle isomorphism.

Note that the 2-skeleton $(\mathbb{R}P^{4n+1})_{[2]}$ is $\mathbb{R}P^2$. We have $\pi_1(SO(n)) \cong \mathbb{Z}_2$ and $\pi_2(SO(n)) = 1$ (for $n > 1$). Hence an oriented isomorphism of real vector bundles on $(\mathbb{R}P^{4n+1})_{[2]}$ is stably homotopic to identity if it is so on the 1-skeleton $S^1 = (\mathbb{R}P^{4n+1})_{[1]}$.

It is easy to see that $c \oplus c$ is homotopic to identity. So it remains to consider $c \oplus -1 : \eta \oplus \mathbb{R} \to \eta \oplus \mathbb{R}$ on S^1. Note that $\eta|_{S^1} = \xi \oplus \xi$ and this bundle is trivial. The splitting corresponds to the basis $(\cos t/2, \sin t/2)$, $(-\sin t/2, \cos t/2)$. (Here $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$.) The map c is given by $c = (1, -1) : \xi \oplus \xi \to \xi \oplus \xi$. So when we identify $\eta \oplus \mathbb{R} \cong \mathbb{R}^3$ on S^1, the isomorphism $c \oplus -1$ is represented by the matrix

$$\begin{aligned}&\begin{pmatrix} \cos t/2 & \sin t/2 & 0 \\ -\sin t/2 & \cos t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos t/2 & -\sin t/2 & 0 \\ \sin t/2 & \cos t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t & 0 \\ -\sin t & -\cos t & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.\end{aligned}$$

This loop represents the nontrivial homotopy class in $\pi_1(SO(3)) \cong \mathbb{Z}_2$. This proves that the involution $\tau^* : \mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}) \to \mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1})$ is nontrivial. Since $\mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}) \cong \mathbb{Z}_2$, the proof of Proposition 3.14 is complete. $\square$

Using the results in this section, we calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,nov}^{\mathbb{Z}}$ (see (6.2) for the definition of $\Lambda_{0,nov}^{\mathbb{Z}}$) in Subsection 6.3, which provides an example such as Floer cohomology is *not* isomorphic to the ordinary cohomology.

3.3. **Orientations on $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;\vec{Q},\vec{P})$.** In this subsection we recall the definitions of the orientations of $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;\vec{Q},\vec{P})$ from Section 8.4 and Subsection 8.10.2 in [FOOO3]. Here L is not necessarily the fixed point set of an anti-symplectic involution τ.

When we discuss the orientation problem, it suffices to consider the regular parts of the moduli spaces. See Remark 3.7 (2). By Theorem 3.6, we have an orientation on $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ for each stable conjugacy class of relative spin structure. Including marked points, we define an orientation on $\mathcal{M}_{k+1,m}^{\mathrm{reg}}(J;\beta)$ by

$$\begin{aligned}&\mathcal{M}_{k+1,m}^{\mathrm{reg}}(J;\beta)\\ =&\left(\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta) \times \partial D_0^2 \times D_1^2 \times \cdots \times D_m^2 \times \partial D_{m+1}^2 \times \cdots \times \partial D_{m+k}^2\right)/PSL(2:\mathbb{R}).\end{aligned}$$

Here the sub-indices in ∂D_0^2 and ∂D_{m+i}^2 (resp. D_j^2) stand for the positions of the marked points z_0 and z_i (resp. z_j^+). (In Subsection 8.10.2 in [FOOO3] we write the above space as $\mathcal{M}_{(1,k),m}(\beta)$.) Strictly speaking, since the marked points are required to be distinct, the left hand side above is not exactly equal to the right hand side but is an open subset. However, when we discuss orientation problem, we sometimes write as above equation when no confusion can occur.

In (2.3), (2.4) we define $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$ by fiber products. Now we equip the right hand sides in (2.3) and (2.4) with the fiber product orientations using Convention 8.2.1 (3) [FOOO3]. However, we do not use the fiber product orientation themselves as the orientations on $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$, but we use the following orientations twisted from the fiber product orientation: We put $\deg P_i = n - \dim P_i$, $\deg Q_j = 2n - \dim Q_j$ for smooth singular simplicies $f_i : P_i \to L$ and $g_j : Q_j \to M$.

Definition 3.15 (Definition 8.4.1 [FOOO3])**.** For given smooth singular simplicies $f_i : P_i \to L$, we define an orientation on $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k)$ by

$$\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k) := (-1)^{\epsilon(P)}\mathcal{M}_{k+1}(J;\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} \left(\prod_{i=1}^k P_i\right),$$

where

$$\epsilon(P) = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.$$

Definition 3.16 (Definition 8.10.2 [FOOO3])**.** For given smooth singular simplices $f_i : P_i \to L$ in L and $g_j : Q_j \to M$ in M, we define

$$\begin{aligned}
&\mathcal{M}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_1,\dots,P_k)\\
:=&(-1)^{\epsilon(P,Q)}\mathcal{M}_{k+1,m}(J;\beta)_{(ev_1^+,\dots,ev_m^+,ev_1,\dots,ev_k)} \times_{g_1\times\dots\times f_k} \left(\prod_{j=1}^m Q_j \times \prod_{i=1}^k P_i\right),
\end{aligned}$$

where

$$\epsilon(P,Q) = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg P_i + ((k+1)(n+1)+1)\sum_{j=1}^{m}\deg Q_j. \tag{3.5}$$

Replacing $\mathcal{M}_{k+1}(J;\beta)$ and $\mathcal{M}_{k+1,m}(J;\beta)$ on the right hand sides of above definitions by $\mathcal{M}^{\rm main}_{k+1}(J;\beta)$ and $\mathcal{M}^{\rm main}_{k+1,m}(J;\beta)$ respectively, we define orientations on the main components $\mathcal{M}^{\rm main}_{k+1}(J;\beta;P_1,\dots,P_k)$ and $\mathcal{M}^{\rm main}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_1,\dots,P_k)$ in the same way.

When we do not consider the fiber product with $g_j : Q_j \to M$, we drop the second term in (3.5). Thus when $m = 0$, the moduli space in Definition 3.16 is nothing but $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k)$ equipped with the orientation given by Definition 3.15.

When we study the map $\tau_*^{\rm main}$ in (1.1), we have to change the ordering of boundary marked points. Later we use the following lemma which describes the behavior of orientations under the change of ordering of boundary marked points:

Lemma 3.17 (Lemma 8.4.3 [FOOO3])**.** *Let σ be the transposition element $(i,i+1)$ in the k-th symmetric group $\mathfrak{S}_k$. $(i = 1,\dots,k-1)$. Then the action of σ on the moduli space $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_i,P_{i+1},\dots,P_k)$ changing the order of marked points induces an orientation preserving isomorphism*

$$\begin{aligned}
\sigma \ :\ &\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_i,P_{i+1},\dots,P_k)\\
\longrightarrow\ &(-1)^{(\deg P_i+1)(\deg P_{i+1}+1)}\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_{i+1},P_i,\dots,P_k).
\end{aligned}$$

4. The induced maps τ_* and $\tau_*^{\rm main}$

Let (M,ω) be a compact, or tame, symplectic manifold and let $\tau : M \to M$ be an anti-symplectic involution, i.e., a map satisfying $\tau^2 = id$ and $\tau^*(\omega) = -\omega$. We also assume that the fixed point set $L = \operatorname{Fix}\tau$ is non-empty, oriented and compact.

Let $\mathcal{J}_\omega$ be the set of all ω compatible almost complex structures and $\mathcal{J}_\omega^\tau$ its subset consisting of τ-anti-invariant almost complex structures J satisfying $\tau_*J = -J$.

Lemma 4.1. *The space $\mathcal{J}_\omega^\tau$ is non-empty and contractible. It becomes an infinite dimensional (Frechet) manifold.*

Proof. For given $J \in \mathcal{J}_\omega^\tau$, its tangent space $T_J\mathcal{J}_\omega^\tau$ consists of sections Y of the bundle $\operatorname{End}(TM)$ whose fiber at $p \in M$ is the space of linear maps $Y : T_pM \to T_pM$ such that

$$YJ + JY = 0, \quad \omega(Yv,w) + \omega(v,Yw) = 0, \quad \tau^*Y = -Y.$$

Note that the second condition means that JY is a symmetric endomorphism with respect to the metric $g_J = \omega(\cdot, J\cdot)$. It immediately follows that $\mathcal{J}_\omega^\tau$ becomes a manifold. The fact that $\mathcal{J}_\omega^\tau$ is non-empty (and contractible) follows from the polar decomposition theorem by choosing a τ-invariant Riemannian metric on M. $\square$

4.1. **The map τ_* and orientation.** We recall that $\Pi(L) = \pi_2(M,L)/\sim$ where $\beta \sim \beta' \in \pi_2(M,L)$ if and only if $\omega(\beta) = \omega(\beta')$ and $\mu_L(\beta) = \mu_L(\beta')$. (See (2.1).) We notice that for each $\beta \in \Pi(L)$, we defined the moduli space $\mathcal{M}(J;\beta)$ as the union

$$\mathcal{M}(J;\beta) = \bigcup_{B\in\pi_2(M,L);[B]=\beta\in\Pi(L)} \mathcal{M}(J;B). \tag{4.1}$$

We put $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$ and $\overline{z}$ denotes the complex conjugate.

Definition 4.2. Let $J \in \mathcal{J}_\omega^\tau$. For J holomorphic curves $w : (D^2, \partial D^2) \to (M,L)$ and $u : S^2 \to M$, we define $\widetilde{w}$, $\widetilde{u}$ by

$$\widetilde{w}(z) = (\tau \circ w)(\overline{z}), \qquad \widetilde{u}(z) = (\tau \circ u)(\overline{z}). \tag{4.2}$$

For $(D^2, w) \in \mathcal{M}^{\mathrm{reg}}(J;\beta)$, $((D^2, \vec{z}, \vec{z}^+), w) \in \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$ we define

$$\tau_*((D^2,w)) = (D^2,\widetilde{w}), \qquad \tau_*(((D^2,\vec{z},\vec{z}^+),w)) = ((D^2,\vec{\overline{z}},\vec{\overline{z}}^+),\widetilde{w}), \tag{4.3}$$

where

$$\vec{\overline{z}} = (\overline{z}_0,\dots,\overline{z}_k), \qquad \vec{\overline{z}}^+ = (\overline{z}_0^+,\dots,\overline{z}_m^+).$$

Remark 4.3. For $\beta = [w]$, we put $\tau_*\beta = [\widetilde{w}]$. Note if $\tau_\sharp : \pi_2(M,L) \to \pi_2(M,L)$ is the natural homomorphism induced by τ, then

$$\tau_*\beta = -\tau_\sharp\beta.$$

This is because $z \mapsto \overline{z}$ is of degree -1. In fact, we have

$$\tau_*(\beta) = \beta$$

in $\Pi(L)$, since τ_* preserves both the symplectic area and the Maslov index.

Lemma 4.4. *The definition (4.3) induces the maps*

$$\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta) \to \mathcal{M}^{\mathrm{reg}}(J;\beta), \quad \tau_* : \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta) \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta),$$

which satisfy $\tau_* \circ \tau_* = \mathrm{id}$.

Proof. If $(w,(z_0,\dots,z_k),(z_1^+,\dots,z_m^+)) \sim (w',(z_0',\dots,z_k'),(z_1'^+,\dots,z_m'^+))$, we have $\varphi \in PSL(2,\mathbb{R}) = \mathrm{Aut}(D^2)$ such that $w' = w\circ\varphi^{-1}$, $z_i' = \varphi(z_i)$, $z_i'^+ = \varphi(z_i^+)$ by definition. We define $\overline{\varphi} : D^2 \to D^2$ by

$$\overline{\varphi}(z) = \overline{(\varphi(\overline{z}))}. \tag{4.4}$$

Then $\overline{\varphi} \in PSL(2,\mathbb{R})$ and $\widetilde{w}' = \widetilde{w}\circ\overline{\varphi}^{-1}$, $\overline{z}_i' = \overline{\varphi}(\overline{z}_i)$, $\overline{z}_i'^+ = \overline{\varphi}(\overline{z}_i^+)$. The property $\tau_*\circ\tau_* = \mathrm{id}$ is straightforward. $\square$

We note that the mapping $\varphi \mapsto \overline{\varphi}$, $PSL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ is orientation preserving.

In Section 3, we have explained that a choice of stable conjugacy class $[(V,\sigma)] \in \mathrm{Spin}(M,L)$ of relative spin structure on L induces an orientation on $\mathcal{M}_{k+1,m}(J;\beta)$ for any given $\beta \in \Pi(L)$. Hereafter we equip $\mathcal{M}_{k+1,m}(J;\beta)$ with this orientation when we regard it as a space with oriented Kuranishi structure. We write as $\mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ when we specify the stable conjugacy class of relative spin structure.

For an anti-symplectic involution τ of (M,ω), we have the pull back $\tau^*[(V,\sigma)]$ of the stable conjugacy class of relative spin structure $[(V,\sigma)]$. See (3.3). Then from the definition of the map τ_* in Lemma 4.4 we obtain the maps

$$\tau_* : \mathcal{M}^{\rm reg}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]},$$
$$\tau_* : \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)^{[(V,\sigma)]}.$$

Here we note that $\tau_* J = -J$ and we use the same τ-anti-symmetric almost complex structure J in both the source and the target spaces of the map τ_*. If $[(V,\sigma)]$ is τ-relatively spin (i.e., $\tau^*[(V,\sigma)] = [(V,\sigma)]$), τ_* defines involutions of $\mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]}$ and $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ with Kuranishi structures. See Subsection 7.2 for the definition of an automorphism of a space with Kuranishi structure. We refer Section A1.3 [FOOO3] for more detailed explanation of the group action on a space with Kuranishi structure.

Theorem 4.5. *Let L be a fixed point set of an anti-symplectic involution τ and $J \in \mathcal{J}^\tau_\omega$. Suppose that L is oriented and carries a relative spin structure (V,σ). Then the map $\tau_* : \mathcal{M}^{\rm reg}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]}$ is orientation preserving if $\mu_L(\beta) \equiv 0 \mod 4$ and is orientation reversing if $\mu_L(\beta) \equiv 2 \mod 4$.*

Corollary 4.6. *Let L be as in Theorem 4.5. In addition, if L carries a τ-relative spin structure $[(V,\sigma)]$, then the map $\tau_* : \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]}$ is orientation preserving if $\mu_L(\beta) \equiv 0 \mod 4$ and is orientation reversing if $\mu_L(\beta) \equiv 2 \mod 4$.*

We prove Theorem 4.5 in Section 5. Here we give a couple of examples.

Example 4.7. (1) Consider the case of $M = \mathbb{C}P^n$, $L = \mathbb{R}P^n$. In this case, each Maslov index $\mu_L(\beta)$ has the form

$$\mu_L(\beta) = \ell_\beta(n+1)$$

where $\beta = \ell_\beta$ times the generator. We know that when n is even L is not orientable, and so we consider only the case where n is odd. On the other hand, when n is odd, L is relatively spin. The class st is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}_2)$. Moreover, we have proved in Proposition 3.14 that $\mathbb{R}P^{4n+3}$ $(n \geq 0)$ is τ-relatively spin, (indeed, $\mathbb{R}P^{4n+3}$ is spin), but $\mathbb{R}P^{4n+1}$ $(n \geq 1)$ is *not* τ-relatively spin. Then using the above formula for the Maslov index, we can conclude from Theorem 4.5 that the map $\tau_* : \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]}$ is always an orientation preserving involution for any τ-relative spin structure $[(V,\sigma)]$ of $\mathbb{R}P^{4n+3}$.

Of course, $\mathbb{R}P^1$ is spin and so τ-relatively spin. The map τ_* is an orientation preserving involution if ℓ_β is even, and an orientation reversing involution if ℓ_β is odd.

(2) Let M be a Calabi-Yau 3-fold and let $L \subset M$ be the set of real points (i.e., the fixed point set of an anti-holomorphic involutive isometry). In this case, L is orientable (because it is a special Lagrangian) and spin (because any orientable 3-manifold is spin). Furthermore $\mu_L(\beta) = 0$ for any $\beta \in \pi_2(M,L)$. Therefore Theorem 4.5 implies that the map $\tau_* : \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\rm reg}(J;\beta)^{[(V,\sigma)]}$ is orientation preserving for any τ-relative spin structure $[(V,\sigma)]$.

We next include marked points. We consider the moduli space $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$.

Proposition 4.8. *The map* $\tau_* : \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ *is orientation preserving if and only if* $\mu_L(\beta)/2 + k + 1 + m$ *is even.*

Proof. Assuming Theorem 4.5, we prove Proposition 4.8. Let us consider the diagram:

$$\begin{tikzcd}
(S^1)^{k+1} \times (D^2)^m \arrow[r, "c"] & (S^1)^{k+1} \times (D^2)^m \\
((S^1)^{k+1} \times (D^2)^m)_0 \arrow[r, "c"] \arrow[u, "\text{inclusion}"] \arrow[d] & ((S^1)^{k+1} \times (D^2)^m)_0 \arrow[u, "\text{inclusion}"'] \arrow[d] \\
\widetilde{\mathcal{M}}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\text{Prop 4.8}"] \arrow[d, "\mathfrak{forget}"'] & \widetilde{\mathcal{M}}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]} \arrow[d, "\mathfrak{forget}"'] \\
\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\text{Thm 4.5}"] & \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}
\end{tikzcd}$$

Diagram 4.1

Here c is defined by

$$c(z_0, z_1, \ldots, z_k, z_1^+, \ldots, z_m^+) = (\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_k, \overline{z}_1^+, \ldots, \overline{z}_m^+)$$

and $\mathfrak{forget}$ are the forgetful maps of marked points. Here we denote by $((S^1)^{k+1} \times (D^2)^m)_0$ the set of all $c(z_0, z_1, \ldots, z_k, z_1^+, \ldots, z_m^+)$ such that $z_i \neq z_j$, $z_i^+ \neq z_j^+$ for $i \neq j$.

Proposition 4.8 then follows from Theorem 4.5 and the fact that the $\mathbb{Z}_2$-action $\varphi \mapsto \overline{\varphi}$ on $PSL(2,\mathbb{R})$ given by (4.4) is orientation preserving. $\square$

We next extend τ_* to the compactification $\mathcal{M}_{k+1,m}(J;\beta)$ of $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$ and define a continuous map

$$\tau_* : \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}.$$

Theorem 4.9. *The map* $\tau_* : \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ *extends to a map* τ_**, denoted by the same symbol:*

$$\tau_* : \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}. \tag{4.5}$$

It preserves orientation if and only if $\mu_L(\beta)/2 + k + 1 + m$ *is even. In particular, if* $[(V,\sigma)]$ *is a* τ*-relative spin structure, it can be regarded as an involution on the space* $\mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ *with Kuranishi structure.*

Proof. We define τ_* by a triple induction over $\omega(\beta) = \int_\beta \omega$, k and m. Namely we define an order on the set of triples (β, k, m) by the relation

$$\omega(\beta') < \omega(\beta). \tag{4.6a}$$

$$\omega(\beta') = \omega(\beta), \quad k' < k. \tag{4.6b}$$

$$\omega(\beta') = \omega(\beta), \quad k' = k, \quad m' < m. \tag{4.6c}$$

We will define the map (4.5) for (β, k, m) under the assumption that the map is already defined for all (β', k', m') smaller than (β, k, m) with respect to this order.

Let $((\Sigma, \vec{z}, \vec{z}^{\,+}), w) \in \mathcal{M}_{k+1,m}(J;\beta)$. We first assume that Σ has a sphere bubble $S^2 \subset \Sigma$. We remove it from Σ to obtain Σ_0. We add one more marked point to Σ_0

at the location where the sphere bubble used to be attached. Then we obtain an element

$$((\Sigma_0, \vec{z}, \vec{z}^{(0)}), w_0) \in \mathcal{M}_{k+1,m+1-\ell}(J;\beta').$$

Here ℓ is the number of marked points on S^2. By the induction hypothesis, τ_* is already defined on $\mathcal{M}_{k+1,m+1-\ell}(J;\beta')$ since $\omega(\beta) > \omega(\beta')$. We denote

$$((\Sigma_0', \vec{z}\,', \vec{z}^{(0)\prime}), w_0') := \tau_*(((\Sigma_0, \vec{z}, \vec{z}^{(0)}), w_0)).$$

We define $v|_{S^2} : S^2 \to M$ by

$$v(z) = w|_{S^2}(\overline{z}).$$

We assume that the nodal point in $\Sigma_0 \cap S^2$ corresponds to $0 \in \mathbb{C} \cup \{\infty\} \cong S^2$. We also map ℓ marked points on S^2 by $z \mapsto \overline{z}$ whose images we denote by $\vec{z}^{(1)} \in S^2$. We then glue $((S^2, \vec{z}^{(1)}), \widetilde{w})$ to $((\Sigma_0', \vec{z}\,', \vec{z}^{(0)\prime}), \widetilde{w})$ at the point $0 \in S^2$ and at the last marked point of $(\Sigma_0, \vec{z}, \vec{z}^{(0)})$ and obtain a curve which is to be the definition of $\tau_*(((\Sigma, \vec{z}, \vec{z}^{\,+}), w))$.

Next suppose that there is no sphere bubble on Σ. Let Σ_0 be the component containing the 0-th marked point. If there is only one irreducible component of Σ, then τ_* is already defined there. So we assume that there is at least one other disc component. Then Σ is a union of Σ_0 and Σ_i for $i = 1, \dots, m$ $(m \geq 1)$. We regard the unique point in $\Sigma_0 \cap \Sigma_i$ for $i = 1, \dots, m$ as marked points of Σ_0. Here each of Σ_i itself is a union of disc components and is connected. We also regard the point in $\Sigma_0 \cap \Sigma_i$ as $0 \in D^2 \cong \mathbb{H} \cup \{\infty\}$ where D^2 is the irreducible component of Σ_i joined to Σ_0, and also as one of the marked points of Σ_i. This defines an element $((\Sigma_i, \vec{z}^{(i)}, \vec{z}^{\,(i)+}), w_{(i)})$ for each $i = 0, \dots, m$. By an easy combinatorics and the induction hypothesis, we can show that τ_* is already constructed on them. Now we define $\tau_*(((\Sigma, \vec{z}, \vec{z}^{\,+}), w))$ by gluing $\tau_*(((\Sigma_i, \vec{z}^{(i)}, \vec{z}^{\,(i)+}), w_{(i)}))$ to Σ_0.

We next consider Kuranishi structure on a neighborhood of $\tau_*(((\Sigma, \vec{z}, \vec{z}^{\,+}), w))$ in $\mathcal{M}_{k+1,m}(J;\beta)$. Observe that the map $u \mapsto \widetilde{u}$ on the moduli space of spheres defined in (4.2) can be regarded as an involution on the space with Kuranishi structure in the same way as in the proof of Theorem 4.5 given in Section 5. Then we can show that (4.5) induces a map of the space with Kuranishi structure by the same induction process as its construction. More precisely, we prove existence of an involution on $\mathcal{M}_{k+1,m}(J;\beta)$ assuming that τ_* induces involutions on $\mathcal{M}_{k+1,m+1-\ell}(J;\beta')$ with Kuranishi structures for all (β', k', m') smaller than (β, k, m).

Let $((\Sigma, \vec{z}, \vec{z}^{\,+}), w) \in \mathcal{M}_{k+1,m}(J;\beta)$. If it is in $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$, we define an involution on its Kuranishi neighborhood in Theorem 4.5. If Σ is not irreducible, then $((\Sigma, \vec{z}, \vec{z}^{\,+}), w)$ is obtained by gluing some elements corresponding to $(\beta', k', m') < (\beta, k, m)$. For each irreducible component, the involution of its Kuranishi neighborhood is constructed by the induction hypothesis. A Kuranishi neighborhood of $((\Sigma, \vec{z}, \vec{z}^{\,+}), w)$ is a fiber product of the Kuranishi neighborhoods of the gluing pieces and the space of the smoothing parameters of the singular points. By definition, our involution obviously commutes with the process to take the fiber product. For the parameter space of smoothing the interior singularities, the action of the involution is the complex conjugation. For the parameter space of smoothing the boundary singularities, the action of involution is trivial. It is easy to see that the analysis we worked out in Section 7.1 [FOOO3] of the gluing is compatible with the involution. Thus τ_* defines an involution on $\mathcal{M}_{k+1,m}(J;\beta)$ with Kuranishi structure.

The statement on the orientation follows from the corresponding statement on $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$ in Proposition 4.8. □

4.2. **The map $\tau_*^{\rm main}$ and orientation.** We next restrict our maps to the main component of $\mathcal{M}_{k+1,m}(J;\beta)$. As we mentioned before, we observe that the induced map $\tau_* : \mathcal{M}_{k+1,m}(J;\beta) \to \mathcal{M}_{k+1,m}(J;\beta)$ does *not* preserve the main component for $k > 1$. On the other hand the assignment given by

$$\begin{aligned} &(w,(z_0,z_1,z_2,\dots,z_{k-1},z_k),(z_1^+,\dots,z_m^+)) \\ &\longmapsto (\widetilde{w},(\overline{z}_0,\overline{z}_k,\overline{z}_{k-1},\dots,\overline{z}_2,\overline{z}_1),(\overline{z}_1^+,\dots,\overline{z}_m^+)) \end{aligned} \tag{4.7}$$

respects the counter-clockwise cyclic order of $S^1 = \partial D^2$ and so preserves the main component, where $\widetilde{w}$ is as in (4.2). Therefore we consider this map instead which we denote by

$$\tau_*^{\rm main} : \mathcal{M}_{k+1,m}^{\rm main}(J;\beta)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}_{k+1,m}^{\rm main}(J;\beta)^{[(V,\sigma)]}. \tag{4.8}$$

We note that for $k = 0, 1$ we have

$$\tau_*^{\rm main} = \tau_*. \tag{4.9}$$

Proposition 4.10. *The map $\tau_*^{\rm main}$ defines a map between the spaces with Kuranishi structures and satisfies $\tau_*^{\rm main} \circ \tau_*^{\rm main} = \mathrm{id}$. In particular, if $[(V,\sigma)]$ is τ-relatively spin, it defines an involution of the space $\mathcal{M}_{k+1,m}^{\rm main}(J;\beta)^{[(V,\sigma)]}$ with Kuranishi structure.*

The proof is the same as the proof of Theorems 4.5 and 4.9 and so omitted.

We now have the following commutative diagram:

$$\begin{array}{ccc}
(S^1)^{k+1} \times (D^2)^m & \xrightarrow{c'} & (S^1)^{k+1} \times (D^2)^m \\
{\scriptstyle \text{inclusion}}\uparrow & & \uparrow{\scriptstyle \text{inclusion}} \\
((S^1)^{k+1} \times (D^2)^m)_{00} & \xrightarrow{c'} & ((S^1)^{k+1} \times (D^2)^m)_{00} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{M}}_{k+1,m}^{\rm main}(J;\beta)^{\tau^*[(V,\sigma)]} & \xrightarrow{\tau_*^{\rm main}} & \widetilde{\mathcal{M}}_{k+1,m}^{\rm main}(J;\beta)^{[(V,\sigma)]} \\
{\scriptstyle \mathfrak{forget}}\downarrow & & \downarrow{\scriptstyle \mathfrak{forget}} \\
\widetilde{\mathcal{M}}(J;\beta)^{\tau^*[(V,\sigma)]} & \xrightarrow{\tau_*} & \widetilde{\mathcal{M}}(J;\beta)^{[(V,\sigma)]}
\end{array}$$

Diagram 4.2

Here c' is defined by

$$c'(z_0,z_1\dots,z_k,z_1^+,\dots,z_m^+) = (\overline{z}_0,\overline{z}_k,\dots,\overline{z}_1,\overline{z}_1^+,\dots,\overline{z}_m^+)$$

and $\mathfrak{forget}$ are the forgetful maps of marked points. In the diagram, $((S^1)^{k+1} \times (D^2)^m)_{00}$ is the open subset of $(S^1)^{k+1} \times (D^2)^m$ consisting of the points such that all z_i's and z_j^+'s are distinct respectively.

Let $\mathrm{Rev}_k : L^{k+1} \to L^{k+1}$ be the map defined by

$$\mathrm{Rev}_k(x_0,x_1,\dots,x_k) = (x_0,x_k,\dots,x_1).$$

It is easy to see that

$$ev \circ \tau_*^{\rm main} = \mathrm{Rev}_k \circ ev. \tag{4.10}$$

We note again that $\mathrm{Rev}_k = \mathrm{id}$ and $\tau_*^{\rm main} = \tau_*$ for $k = 0, 1$.

Let $P_1, \dots, P_k$ be smooth singular simplices on L. By taking the fiber product and using (4.7), we obtain a map

$$\tau_*^{\rm main} : \mathcal{M}_{k+1,m}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_{k+1,m}^{\rm main}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]} \quad (4.11)$$

which satisfies $\tau_*^{\rm main} \circ \tau_*^{\rm main} = {\rm id}$. We put

$$\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + m + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j. \quad (4.12)$$

Theorem 4.11. *The map (4.11) preserves orientation if ϵ is even, and reverses orientation if ϵ is odd.*

The proof of Theorem 4.11 is given in Section 5.

5. Proofs of Theorem 4.5 and Theorem 4.11

In this section we prove Theorem 4.5 (= Theorem 1.1) and Theorem 4.11 (= Theorem 1.4) stated in the previous section.

Proof of Theorem 4.5. Pick $J \in \mathcal{J}_\omega^\tau$, a τ-anti-invariant almost complex structure compatible with ω. For a J holomorphic curve $w : (D^2, \partial D^2) \to (M, L)$, we recall that we define $\widetilde{w}$ by

$$\widetilde{w}(z) = (\tau \circ w)(\overline{z}).$$

Moreover for $(D^2, w) \in \mathcal{M}^{\rm reg}(J;\beta)$, $((D^2, \vec{z}, \vec{z}^+), w) \in \mathcal{M}_{k+1,m}^{\rm reg}(J;\beta)$ we define

$$\tau_*((D^2, w)) = (D^2, \widetilde{w}), \qquad \tau_*(((D^2, \vec{z}, \vec{z}^+), w)) = ((D^2, \vec{\overline{z}}, \vec{\overline{z}}^+), \widetilde{w}),$$

where

$$\vec{\overline{z}} = (\overline{z}_0, \dots, \overline{z}_k), \qquad \vec{\overline{z}}^+ = (\overline{z}_0^+, \dots, \overline{z}_m^+).$$

Let $[D^2, w] \in \mathcal{M}^{\rm reg}(J;\beta)$. We consider the deformation complex

$$D_w\overline{\partial} : \Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \longrightarrow \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \quad (5.1)$$

and

$$D_{\widetilde{w}}\overline{\partial} : \Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \longrightarrow \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM), \quad (5.2)$$

where $D_w\overline{\partial}$ is the linearized operator of pseudo-holomorphic curve equation as (3.1). (Here and hereafter, $\Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ is the decomposition of the complexified cotangent bundle of the *domain* of pseudo-holomorphic curves.

We have the commutative diagram

$$\begin{tikzcd}
(w^*TM, w|_{\partial D^2}^* TL) \arrow[r, "T\tau"] \arrow[d] & (\widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \arrow[d] \\
(D^2, \partial D^2) \arrow[r, "c"] & (D^2, \partial D^2)
\end{tikzcd}$$

Diagram 5.1

where $c(z) = \overline{z}$ and we denote by $T\tau$ the differential of τ. It induces a bundle map

$$\mathrm{Hom}_{\mathbb{R}}(TD^2, w^*TM) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(TD^2, \widetilde{w}^*TM), \quad (5.3)$$

which covers $z \mapsto \overline{z}$. The bundle map (5.3) is fiberwise anti-complex linear i.e.,

$$\mathrm{Hom}_{\mathbb{R}}(T_zD^2, T_{w(z)}M) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(T_{\overline{z}}D^2, T_{\tau(w(z))}M),$$

is anti-complex linear at each $z \in D^2$. Therefore it preserves the decomposition

$$\mathrm{Hom}_{\mathbb{R}}(TD^2, w^*TM) \otimes \mathbb{C} = (\Lambda^{1,0} \otimes w^*TM) \oplus (\Lambda^{0,1} \otimes w^*TM), \tag{5.4}$$

since (5.4) is the decomposition to the complex and anti-complex linear parts. Hence we obtain a map

$$(T_{w,1}\tau)_* : \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \longrightarrow \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM)$$

which is anti-complex linear. In the similar way, we obtain an anti-complex linear map:

$$(T_{w,0}\tau)_* : \Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \longrightarrow \Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL).$$

Since τ is an isometry, it commutes with the covariant derivative. This gives rise to the following commutative diagram.

$$\begin{tikzcd}
\Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \arrow[r, "D_w\overline{\partial}"] \arrow[d, "(T_{w,0}\tau)_*"'] & \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \arrow[d, "(T_{w,1}\tau)_*"'] \\
\Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \arrow[r, "D_{\widetilde{w}}\overline{\partial}"] & \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM)
\end{tikzcd}$$

Diagram 5.2

To define a Kuranishi chart in a neighborhood of $[D^2, w]$ we need to take a finite dimensional subspace $E_{[D^2,w]}$ of $\Gamma(D^2; \Lambda^{0,1} \otimes w^*TM)$ such that

$$\mathrm{Im}\, D_w\overline{\partial} + E_{[D^2,w]} = \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM).$$

We choose $E_{[D^2,w]}$ so that it is invariant under $(T_{w,1}\tau)_*$, i.e.,

$$E_{[D^2,\widetilde{w}]} = (T_{w,1}\tau)_*(E_{[D^2,w]}).$$

Let $w' : (D^2, \partial D^2) \to (M, L)$ be a map C^0-close to w. By definition it is easy to see that

$$\overline{\partial}\widetilde{w}' = (T_{w',1}\tau)_*(\overline{\partial} w').$$

We can take an isomorphism

$$I_{w,w'} : \Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \cong \Gamma(D^2, \partial D^2 : (w')^*TM, w'|_{\partial D^2}^* TL)$$

that is complex linear and satisfies

$$(T_{w',0}\tau)_* \circ I_{w,w'} = I_{\widetilde{w},\widetilde{w}'} \circ (T_{w,0}\tau)_*. \tag{5.5}$$

Now we recall that a Kuranishi neighborhood $V_{[D^2,w]}$ is constructed in Section 7.1 in [FOOO3] which is given by the set of solutions of the equation

$$\overline{\partial}\widetilde{w}' \equiv 0 \mod I_{w,w'}(E_{[D^2,w]}).$$

Hence by (5.5) the map $w' \mapsto \widetilde{w}'$ defines a diffeomorphism

$$\tau_* : V_{[D^2,w]} \cong V_{[D^2,\widetilde{w}]}.$$

Moreover the Kuranishi map $w' \mapsto s(w') = \overline{\partial} w'$ commutes with τ_*. Therefore τ_* induces an involution of the Kuranishi structure.

We next study the orientation. Let $w \in \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ and consider $\widetilde{w} \in \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$. We consider commutative Diagram 5.1. A trivialization

$$\Phi : (w^*TM, w|_{\partial D^2}^* TL) \longrightarrow (D^2, \partial D^2; \mathbb{C}^n, \Lambda)$$

naturally induces a trivialization

$$\widetilde{\Phi} : (\widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \longrightarrow (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}),$$

where $\Lambda : S^1 \simeq \partial D^2 \to \Lambda(\mathbb{C}^n)$ is a loop of Lagrangian subspaces given by $\Lambda(z) := T_{w(z)}L$ in the trivialization and $\widetilde{\Lambda}$ is defined by

$$\widetilde{\Lambda}(z) = \Lambda(\overline{z}).$$

Note that if the Lagrangian loop $\widetilde{\Lambda}$ represents the homotopy class of a trivialization induced by the relative spin structure $[(V,\sigma)]$, Λ will represent the homotopy class of a trivialization induced by $\tau^*[(V,\sigma)]$. With respect to these trivializations, we have the commutative diagram

$$\begin{tikzcd}
(D^2, \partial D^2; \mathbb{C}^n, \Lambda) \arrow[r, "\widetilde{\Phi}\circ T\tau\circ\Phi^{-1}"] \arrow[d] & (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}) \arrow[d] \\
(D^2, \partial D^2) \arrow[r, "c"] & (D^2, \partial D^2)
\end{tikzcd}$$

Diagram 5.3

and the elliptic complex (5.1), (5.2). Note that the map

$$\tau_z := \widetilde{\Phi} \circ T\tau \circ \Phi^{-1}(z, \cdot) : (\mathbb{C}^n, \Lambda(z)) \longrightarrow (\mathbb{C}^n, \Lambda(z))$$

defines an involution with the Lagrangian subspace $\Lambda(z)$ fixed. Now we deform the metric on D^2 and the trivialization Φ so that $\Lambda(z) \equiv \mathbb{R}^n$. Recall that we assume L is orientable and so the bundle $w|_{\partial D^2}^* TL \to S^1$ is trivial. We also recall the argument explained in Remark 3.9. Namely, after deforming further the Cauchy-Riemann operator on $(D^2, \partial D^2; \mathbb{C}^n, \Lambda)$, we are reduced to considering the case

$$\begin{aligned} C : \operatorname{Ker} \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)} &\times \text{Hol } (\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \\ &\longrightarrow \operatorname{Ker} \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)} \times \text{Hol } (\mathbb{C}P^1; E'|_{\mathbb{C}P^1}). \end{aligned} \tag{5.6}$$

Here E' is a complex vector bundle over the nodal curve $\Sigma = D^2 \cup \mathbb{C}P^1$ with a nodal point $p = O$. The topology of the bundle $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$ is determined by the loop Λ of Lagrangian subspaces defined by $\Lambda(z) = T_{w(z)}L$ in the trivialization. Therefore we have

$$\dim_{\mathbb{C}} \text{Hol } (\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) = \frac{1}{2}\mu(\Lambda).$$

The map C in (5.6) is the natural map induced from the conjugation on $\mathbb{C}^n$ and E'. The first factor in (5.6) is invariant under the conjugation because

$$\operatorname{Ker} \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)} \cong \mathbb{R}^n$$

from (3.2). For the second factor, we have Hol $(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \cong \mathbb{C}^{\frac{1}{2}\mu(\Lambda)}$ with C is reduced to the standard conjugation on $\mathbb{C}^{\frac{1}{2}\mu(\Lambda)}$. Therefore it boils down to considering the conjugation

$$C : \mathbb{C}^{\frac{1}{2}\mu(\Lambda)} \longrightarrow \mathbb{C}^{\frac{1}{2}\mu(\Lambda)}.$$

It is easy to see that this map is orientation preserving if and only if $\frac{1}{2}\mu(\Lambda) \equiv 0 \mod 2$, i.e., $\mu(\Lambda) \equiv 0 \mod 4$. We note, by definition, that $\mu_L(\beta) = \mu(\Lambda)$. This finishes the proof of Theorem 4.5. □

Proof of Theorem 4.11. To prove the assertion on orientation, it is enough to consider the orientation on the regular part $\mathcal{M}^{\text{main, reg}}_{k+1,m}(J;\beta;P_1,\dots,P_k)$. See Remark 3.7 (2). By Theorem 4.5, $\tau_* : \mathcal{M}^{\text{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\text{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation preserving if and only if $\mu_L(\beta)/2$ is even. Recall when we consider the main component $\mathcal{M}^{\text{main,reg}}_{k+1,m}(J;\beta)$, the boundary marked points is in counter-clockwise cyclic ordering. However, by the involution τ_* in Theorem 4.11, each boundary marked point z_i is mapped to $\overline{z}_i$ and each interior marked point z^+_j is mapped to $\overline{z^+_j}$. Thus the order of the boundary marked points changes to clockwise ordering. Denote by $\mathcal{M}^{\text{clock, reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ the moduli space with the boundary marked points $(z_0,z_1,\dots,z_k)$ respect the *clock-wise* orientation and interior marked points $z^+_1,\dots,z^+_m$. Since $z\mapsto\overline{z}$ reverses the orientation on the boundary and $z^+\mapsto\overline{z^+}$ reverses the orientation on the interior, the argument in the proof of Lemma 4.8 shows that $\tau_* : \mathcal{M}^{\text{main, reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\text{clock, reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ respects the orientation if and only if $\mu_L(\beta)/2+k+1+m$ is even. Thus we have

$$
\begin{aligned}
&\mathcal{M}^{\text{main, reg}}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]}\\
=&(-1)^{\mu_L(\beta)/2+k+1+m}\mathcal{M}^{\text{clock, reg}}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{[(V,\sigma)]}.
\end{aligned}
$$

Recall that Lemma 3.17 describes how the orientation of $\mathcal{M}_{k+1,m}(J;\beta;P_1,\dots,P_k)$ changes by changing ordering of boundary marked points. Thus, using Lemma 3.17, we obtain Theorem 4.11 immediately. □

Since the map τ^{main}_* preserves the ordering of interior marked points, we also obtain the following:

Theorem 5.1. *Let $Q_1,\dots,Q_m$ be smooth singular simplicies of M. Then the map*

$$
\begin{aligned}
\tau^{\text{main}}_* \ : \ &\mathcal{M}^{\text{main}}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]}\\
&\longrightarrow \mathcal{M}^{\text{main}}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_k,\dots,P_1)^{[(V,\sigma)]}
\end{aligned}
$$

preserves orientation if and only if

$$
\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + m + \sum_{1\le i<j\le k} \deg' P_i \deg' P_j
$$

is even.

6. Applications

Using results obtained in the previous sections, we prove Theorem 1.5, Corollary 1.6, Corollary 1.8, Theorem 1.9 and Corollary 1.10 in Subsection 6.2 and calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda^{\mathbb{Z}}_{0,nov}$ in Subsection 6.3.

6.1. **Filtered A_∞ algebra and Lagrangian Floer cohomology.**

In order to prove applications to Lagrangian Floer theory, we very quickly recall necessary material of the filtered A_∞ algebra constructed from a relatively spin closed Lagrangian submanifold L and obstruction/deformation theory of Lagrangian Floer cohomology developed in our book [FOOO3]. See a survey paper [Oht] for a more detailed review.

Let R be a commutative ring with unit. Let e and T be formal variables of degree 2 and 0, respectively. We use the *universal Novikov ring* over R as our coefficient ring:

$$\Lambda_{nov}^R = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \;\middle|\; a_i \in R,\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lim_{\imath \to \infty} \lambda_i = +\infty \right\}, \tag{6.1}$$

$$\Lambda_{0,nov}^R = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{nov} \;\middle|\; \lambda_i \geq 0 \right\}. \tag{6.2}$$

We define a filtration $F^\lambda \Lambda_{0,nov}^R = T^\lambda \Lambda_{0,nov}^R$ ($\lambda \in \mathbb{R}_{\geq 0}$) on $\Lambda_{0,nov}^R$ which induces a filtration $F^\lambda \Lambda_{nov}^R$ ($\lambda \in \mathbb{R}$) on Λ_{nov}^R. We call it *energy filtration*. Given these filtrations, both $\Lambda_{0,nov}^R$ and Λ_{nov}^R become graded filtered commutative rings. In the rest of this subsection and the next, we take $R = \mathbb{Q}$. We use the case $R = \mathbb{Z}$ in Subsection 6.3.

In Section 2 we define $\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)$ for smooth singular simplicies (P_i,f_i) of L. By the result of Section 7.1 [FOOO3] it has a Kuranishi structure. Here we use the same notations about Kuranishi structure as the ones used in Appendix of the present paper. The space $\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)$ is locally described by $s_p^{-1}(0)/\Gamma_p$. If the Kuranishi map s_p is transverse to the zero section, it is locally an orbifold. However, so far as Γ_p is non trivial, we can not perturb s_p to a Γ_p-equivariant section transverse to the zero section, in general. Instead of single valued sections, we take a Γ_p-equivariant *muti-valued* section (multi-section) $\mathfrak{s}_p$ of $E_p \to V_p$ so that each branch is transverse to the zero section and $\mathfrak{s}_p^{-1}(0)/\Gamma_p$ and sufficiently close to $s_p^{-1}(0)/\Gamma_p$. (See Sections 7.1 and 7.2 in [FOOO3] for the precise statement.) We denote the perturbed zero locus (divided by Γ_p) by $\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}$. We have the evaluation map at the zero-th marked point for the perturbed moduli space:

$$ev_0 : \mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}} \longrightarrow L.$$

Then such a system $\mathfrak{s} = \{\mathfrak{s}_p\}$ of muti-valued sections gives rise to the virtual fundamental chain over $\mathbb{Q}$ as follows: By Lemma 6.9 in [FO] and Lemma A1.26 in [FOOO3], $\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}$ has a smooth triangulation. We take a smooth triangulation on it. Each simplex Δ_a^d of dimension $d = \dim \mathfrak{s}_p^{-1}(0)$ in the triangulation comes with multiplicity $\mathrm{mul}_{\Delta_a^d} \in \mathbb{Q}$. (See Definition A1.27 in [FOOO3] for the definition of multiplicity.) Restricting ev_0 to Δ_a^d, we have a singular simplex of L denoted by (Δ_a^d, ev_0). Then the virtual fundamental chain over $\mathbb{Q}$ which we denote by $(\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, ev_0)$ is defined by

$$(\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, ev_0) = \sum_a \mathrm{mul}_{\Delta_a^d} \cdot (\Delta_a^d, ev_0).$$

When the virtual dimension is zero, i.e. when $d = 0$, we denote

$$\#\mathcal{M}_{k+1}^{\rm main}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}} = \sum_a \mathrm{mul}_{\Delta_a^0} \in \mathbb{Q}.$$

Now put $\Pi(L)_0 = \{(\omega(\beta),\mu_L(\beta)) \mid \beta \in \Pi(L),\ \mathcal{M}(J;\beta) \neq \emptyset\}$, see (2.1) for $\Pi(L)$. Let $G(L)$ be a submonoid of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ generated by $\Pi(L)_0$. We put $\beta_0 = (0,0) \in G(L)$.

Definition 6.1. For smooth singular simplicies P_i of L and $\beta \in G(L)$, we define a series of maps $\mathfrak{m}_{k,\beta}$ by

$$\mathfrak{m}_{0,\beta}(1) = \begin{cases} (\mathcal{M}_1(J;\beta)^{\mathfrak{s}}, ev_0) & \text{for } \beta \neq \beta_0 \\ 0 & \text{for } \beta = \beta_0, \end{cases}$$

$$\mathfrak{m}_{1,\beta}(P) = \begin{cases} (\mathcal{M}_2^{\text{main}}(J;\beta;P)^{\mathfrak{s}}, ev_0) & \text{for } \beta \neq \beta_0 \\ (-1)^n \partial P & \text{for } \beta = \beta_0, \end{cases}$$

and

$$\mathfrak{m}_{k,\beta}(P_1,\dots,P_k) = (\mathcal{M}_{k+1}^{\text{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, ev_0), \quad k \geq 2.$$

Here ∂ is the usual boundary operator and $n = \dim L$. Then, one of main results proved in [FOOO3] is as follows: For a smooth singular chain P on L we put the cohomological grading $\deg P = n - \dim P$ and regard a smooth singular chain complex $S_*(L;\mathbb{Q})$ as a smooth singular cochain complex $S^{n-*}(L;\mathbb{Q})$. For a subcomplex $C(L;\mathbb{Q})$ of $S(L;\mathbb{Q})$ we denote by $C(L;\Lambda_{0,nov}^{\mathbb{Q}})$ a completion of $C(L;\mathbb{Q})\otimes \Lambda_{0,nov}^{\mathbb{Q}}$ with respect to the filtration induced from one on $\Lambda_{0,nov}^{\mathbb{Q}}$ introduced above. We shift the degree by 1 i.e., define

$$C(L;\Lambda_{0,nov}^{\mathbb{Q}})[1]^{\bullet} = C(L;\Lambda_{0,nov}^{\mathbb{Q}})^{\bullet+1},$$

where we define $\deg(PT^{\lambda}e^{\mu}) = \deg P + 2\mu$ for $PT^{\lambda}e^{\mu} \in C(L;\Lambda_{0,nov}^{\mathbb{Q}})$. We put

$$\mathfrak{m}_k = \sum_{\beta \in G(L)} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)} e^{\mu_L(\beta)/2}, \quad k = 0,1,\dots. \tag{6.3}$$

To simplify notation we write $C = C(L;\Lambda_{0,nov}^{\mathbb{Q}})$. Put

$$B_k(C[1]) = \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k}$$

and take its completion with respect to the energy filtration. By an abuse of notation, we denote the completion by the same symbol. We define the *bar complex* $B(C[1]) = \bigoplus_{k=0}^{\infty} B_k(C[1])$ and extend $\mathfrak{m}_k$ to the graded coderivation $\widehat{\mathfrak{m}}_k$ on $B(C[1])$ by

$$\widehat{\mathfrak{m}}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{k-i+1} (-1)^* x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i,\dots,x_{i+k-1}) \otimes \cdots \otimes x_n \tag{6.4}$$

where $* = \deg x_1 + \cdots + \deg x_{i-1} + i - 1$. We put

$$\widehat{d} = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k : B(C[1]) \longrightarrow B(C[1]). \tag{6.5}$$

Theorem 6.2 (Theorem 3.5.11 in [FOOO3]). *For any closed relatively spin Lagrangian submanifold L of M, there exist a countably generated subcomplex $C(L;\mathbb{Q})$ of smooth singular cochain complex of L whose cohomology is isomorphic to $H(L;\mathbb{Q})$ and a system of multi-sections $\mathfrak{s}$ of $\mathcal{M}_{k+1}^{\text{main}}(J;\beta;P_1,\dots,P_k)$ ($\mathfrak{s}$ are chosen depending on $P_i \in C(L;\mathbb{Q})$) such that*

$$\mathfrak{m}_k : \underbrace{C(L;\Lambda_{0,nov}^{\mathbb{Q}})[1] \otimes \cdots \otimes C(L;\Lambda_{0,nov}^{\mathbb{Q}})[1]}_{k} \to C(L;\Lambda_{0,nov}^{\mathbb{Q}})[1], \quad k = 0,1,\dots$$

are defined and satisfy $\widehat{d} \circ \widehat{d} = 0$.

The equation $\widehat{d} \circ \widehat{d} = 0$ is equivalent to

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{\deg' x_1 + \cdots + \deg' x_{i-1}} \mathfrak{m}_{k_1}(x_1, \ldots, \mathfrak{m}_{k_2}(x_i, \ldots, x_{i+k_2-1}), \ldots, x_k) = 0$$

which we call the A_∞ *formulas*. Here $\deg' x_i = \deg x_i - 1$, the shifted degree. In particular, the A_∞ formulas imply an equality

$$\mathfrak{m}_2(\mathfrak{m}_0(1), x) + (-1)^{\deg' x} \mathfrak{m}_2(x, \mathfrak{m}_0(1)) + \mathfrak{m}_1\mathfrak{m}_1(x) = 0,$$

which shows $\mathfrak{m}_1 \circ \mathfrak{m}_1 \neq 0$ if $\mathfrak{m}_0(1) \neq 0$, in general. So $\mathfrak{m}_0$ gives an obstruction to define $\mathfrak{m}_1$-cohomology.

Definition 6.3. An element $b \in C(L; \Lambda^{\mathbb{Q}}_{0,nov})[1]^0$ with $b \equiv 0 \mod \Lambda^+_{0,nov}$ is called a *solution of the Maurer-Cartan equation* or *bounding cochain*, if it satisfies the Maurer-Cartan equation:

$$\mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b, b) + \mathfrak{m}_3(b, b, b) + \cdots = 0.$$

Here $\Lambda^+_{0,nov} = \{\sum a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{0,nov} \mid \lambda_i > 0\}$. We denote by $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov})$ the set of all bounding cochains. We say L is *unobstructed* if $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov}) \neq \emptyset$.

Remark 6.4. We do not introduce the notion of gauge equivalence of bounding cochains (Definition 4.3.1 in [FOOO3]), because we do not use it in this paper.

If $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov}) \neq \emptyset$, then by using any $b \in \mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov})$ we can deform the A_∞ structure $\mathfrak{m}$ to $\mathfrak{m}^b$ by

$$\mathfrak{m}^b_k(x_1, \ldots, x_k) = \sum_{\ell_0, \ldots, \ell_k} \mathfrak{m}_{k+\sum \ell_i}(\underbrace{b, \ldots, b}_{\ell_0}, x_1, \underbrace{b, \ldots, b}_{\ell_1}, \ldots, \underbrace{b, \ldots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \ldots, b}_{\ell_k})$$

so that $\mathfrak{m}^b_1 \circ \mathfrak{m}^b_1 = 0$ (Proposition 3.6.10 in [FOOO3]). Then we can define

$$HF((L, b); \Lambda^{\mathbb{Q}}_{0,nov}) := H(C(L; \Lambda^{\mathbb{Q}}_{0,nov}), \mathfrak{m}^b_1)$$

which we call *Floer cohomology of* L (deformed by b).

Next, let $(L^{(1)}, L^{(0)})$ be a relatively spin pair of closed Lagrangian submanifolds. We first assume that $L^{(0)}$ is transverse to $L^{(1)}$. Let $C(L^{(1)}, L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov})$ be the free $\Lambda^{\mathbb{Q}}_{0,nov}$ module generated by the intersection points $L^{(1)} \cap L^{(0)}$. Then we can construct a filtered A_∞ bimodule structure $\{\mathfrak{n}_{k_1,k_0}\}_{k_1,k_0=0,1,\ldots}$ on $C(L^{(1)}, L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov})$ over the pair $(C(L^{(1)}; \Lambda^{\mathbb{Q}}_{0,nov}), C(L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov}))$ of A_∞ algebras as follows. Here we briefly describe the map

$$\begin{aligned}\mathfrak{n}_{k_1,k_0} : B_{k_1}(C(L^{(1)}; \Lambda^{\mathbb{Q}}_{0,nov})[1]) \otimes C(L^{(1)}, L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov}) \otimes B_{k_0}(C(L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov})[1]) \\ \longrightarrow C(L^{(1)}, L^{(0)}; \Lambda^{\mathbb{Q}}_{0,nov}).\end{aligned}$$

A typical element of the tensor product above is written as

$$\left(P^{(1)}_1 T^{\lambda^{(1)}_1} e^{\mu^{(1)}_1} \otimes \cdots \otimes P^{(1)}_{k_1} T^{\lambda^{(1)}_{k_1}} e^{\mu^{(1)}_{k_1}}\right) \otimes T^\lambda e^\mu \langle p \rangle \otimes \left(P^{(0)}_1 T^{\lambda^{(0)}_1} e^{\mu^{(0)}_1} \otimes \cdots \otimes P^{(0)}_{k_0} T^{\lambda^{(0)}_{k_0}} e^{\mu^{(0)}_{k_0}}\right)$$

for $p \in L^{(1)} \cap L^{(0)}$. Then $\mathfrak{n}_{k_1,k_0}$ maps it to

$$\sum_{q,B} \# \left(\mathcal{M}(p, q; B; P^{(1)}_1, \ldots, P^{(1)}_{k_1}; P^{(0)}_1, \ldots, P^{(0)}_{k_0})\right) T^{\lambda'} e^{\mu'} \langle q \rangle$$

with $\lambda' = \omega(B) + \sum \lambda^{(1)}_i + \lambda + \sum \lambda^{(0)}_i$ and $\mu' = \mu_L(B) + \sum \mu^{(1)}_i + \mu + \sum \mu^{(0)}_i$. Here B is the homotopy class of Floer trajectories connecting p and q, and the

sum is taken over all (q, B) such that the virtual dimension of the moduli space $\mathcal{M}(p, q; B; P_1^{(1)}, \dots, P_{k_1}^{(1)}; P_1^{(0)}, \dots, P_{k_0}^{(0)})$ of Floer trajectories is zero. See Subsection 3.7.4 of [FOOO3] for the precise definition of

$$\mathcal{M}(p, q; B; P_1^{(1)}, \dots, P_{k_1}^{(1)}; P_1^{(0)}, \dots, P_{k_0}^{(0)}).$$

Strictly speaking, we also need to take a suitable system of multi-sections on this moduli space to obtain the virtual fundamental chain that enters in the construction of the operators $\mathfrak{n}_{k_1,k_0}$ defining the desired A_∞ bimodule structure. Because of the usage of multi-sections, the counting number with sign

$$\#\left(\mathcal{M}(p, q; B; P_1^{(1)}, \dots, P_{k_1}^{(1)}; P_1^{(0)}, \dots, P_{k_0}^{(0)})\right)$$

is a rational number, in general.

Now let $B(C(L^{(1)}; \Lambda_{0,nov}^{\mathbb{Q}})[1]) \otimes C(L^{(1)}, L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}}) \otimes B(C(L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}})[1])$ be the completion of

$$\bigoplus_{k_0 \geq 0, k_1 \geq 0} B_{k_1}(C(L^{(1)}; \Lambda_{0,nov}^{\mathbb{Q}})[1]) \otimes C(L^{(1)}, L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}}) \otimes B_{k_0}(C(L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}})[1])$$

with respect to the induced energy filtration. We extend $\mathfrak{n}_{k_1,k_0}$ to a bi-coderivation on $B(C(L^{(1)}; \Lambda_{0,nov}^{\mathbb{Q}})[1]) \otimes C(L^{(1)}, L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}}) \otimes B(C(L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}})[1])$ which is given by the formula

$$\begin{aligned}
&\widehat{d}(x_1^{(1)} \otimes \cdots \otimes x_{k_1}^{(1)} \otimes y \otimes x_1^{(0)} \otimes \cdots \otimes x_{k_0}^{(0)}) \\
= &\sum_{k_1' \leq k_1, k_0' \leq k_0} (-1)^{\deg' x_1^{(1)} + \cdots + \deg' x_{k_1 - k_1'}^{(1)}} \\
&\quad x_1^{(1)} \otimes \cdots \otimes x_{k_1-k_1'}^{(1)} \otimes \mathfrak{n}_{k_1',k_0'}(x_{k_1-k_1'+1}^{(1)}, \cdots, y, \cdots, x_{k_0'}^{(0)}) \otimes x_{k_0'+1}^{(0)} \otimes \cdots \otimes x_{k_0}^{(0)} \\
&\quad + \widehat{d}^{(1)}(x_1^{(1)} \otimes \cdots \otimes x_{k_1}^{(1)}) \otimes y \otimes x_1^{(0)} \otimes \cdots \otimes x_{k_0}^{(0)} \\
&\quad + (-1)^{\Sigma \deg' x_i^{(1)} + \deg' y} x_1^{(1)} \otimes \cdots \otimes x_{k_1}^{(1)} \otimes y \otimes \widehat{d}^{(0)}(x_1^{(0)} \otimes \cdots \otimes x_{k_0}^{(0)}).
\end{aligned} \tag{6.6}$$

Here $\widehat{d}^{(i)}$ is defined by (6.4) and (6.5), using the filtered A_∞ structure $\mathfrak{m}^{(i)}$ of $(C(L^{(i)}; \Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{m}^{(i)})$ $(i = 0, 1)$.

Theorem 6.5 (Theorem 3.7.21 in [FOOO3]). *For any relatively spin pair $(L^{(1)}, L^{(0)})$ of closed Lagrangian submanifolds, the family of maps $\{\mathfrak{n}_{k_1,k_0}\}_{k_1,k_0}$ defines a filtered A_∞ bimodule structure on $C(L^{(1)}, L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}})$ over $(C(L^{(1)}; \Lambda_{0,nov}^{\mathbb{Q}}), C(L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}}))$. Namely, $\widehat{d}$ in (6.6) satisfies $\widehat{d} \circ \widehat{d} = 0$.*

Since the equation $\widehat{d} \circ \widehat{d} = 0$ implies, in particular,

$$\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0}(y) + \mathfrak{n}_{1,0}(\mathfrak{m}_0^{(1)}(1), y) + (-1)^{\deg' y} \mathfrak{n}_{0,1}(y, \mathfrak{m}_0^{(0)}(1)) = 0,$$

we have $\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0} \neq 0$, in general. However, if both of $L^{(0)}$ and $L^{(1)}$ are unobstructed in the sense of Definition 6.3, we can deform the filtered A_∞ bimodule structure $\mathfrak{n}$ by $b_i \in \mathcal{M}(L^{(i)}; \Lambda_{0,nov}^{\mathbb{Q}})$ so that

$${}^{b_1}\mathfrak{n}_{0,0}^{b_0}(y) := \sum_{k_1,k_0} \mathfrak{n}_{k_1,k_0}(\underbrace{b_1, \dots, b_1}_{k_1}, y, \underbrace{b_0, \dots, b_0}_{k_0})$$

satisfies ${}^{b_1}\mathfrak{n}_{0,0}^{b_0} \circ {}^{b_1}\mathfrak{n}_{0,0}^{b_0} = 0$ (Lemma 3.7.14 in [FOOO3]). Then we can define

$$HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,nov}^{\mathbb{Q}}) := H(C(L^{(1)}, L^{(0)}; \Lambda_{0,nov}^{\mathbb{Q}}), {}^{b_1}\mathfrak{n}_{0,0}^{b_0})$$

which we call *Floer cohomology of a pair* $(L^{(1)}, L^{(0)})$ (deformed by b_1, b_0).

So far we assume that $L^{(0)}$ is transverse to $L^{(1)}$. But we can generalize the story to the Bott-Morse case, that is, each component of $L^{(0)} \cap L^{(1)}$ is a smooth manifold. Especially, for the case $L^{(1)} = L^{(0)}$, we have $\mathfrak{n}_{k_1,k_0} = \mathfrak{m}_{k_1+k_0+1}$ (see Example 3.7.6 in [FOOO3]) and an isomorphism

$$HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \cong HF((L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \tag{6.7}$$

for any $b \in \mathcal{M}(L; \Lambda_{0,nov}^{\mathbb{Q}})$ by Theorem G (G.1) in [FOOO3]. Moreover, if we extend the coefficient ring $\Lambda_{0,nov}^{\mathbb{Q}}$ to $\Lambda_{nov}^{\mathbb{Q}}$, we can find that Hamiltonian isotopies $\Psi_i^s : M \to M$ ($i = 0, 1, s \in [0, 1]$) with $\Psi_i^0 = \mathrm{id}$ and $\Psi_i^1 = \Psi_i$ induce an isomorphism

$$\begin{aligned} &HF((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{nov}^{\mathbb{Q}}) \\ \cong\ &HF((\Psi_1(L^{(1)}), \Psi_{1*}b_1), (\Psi_0(L^{(0)}), \Psi_{0*}b_0); \Lambda_{nov}^{\mathbb{Q}}) \end{aligned} \tag{6.8}$$

by Theorem G (G.4) in [FOOO3]. This shows invariance of Floer cohomology of a pair $(L^{(1)}, L^{(0)})$ over $\Lambda_{nov}^{\mathbb{Q}}$ under Hamiltonian isotopies.

6.2. **Proofs of applications.**

Proof of Theorem 1.5. We consider the map (4.11) for the case $m = 0$. It is an automorphism of order 2. We first take its quotient by Lemma 7.6 (Lemma A1.49 [FOOO3]) in the sense of Kuranishi structure, and take a perturbed multi-section of the quotient space, which is transverse to zero section. After that we lift the perturbed multi-section. Then we can obtain a system of multi-sections $\mathfrak{s}$ on the moduli space $\mathcal{M}_{k+1}^{\mathrm{main}}(J; \beta; P_1, \ldots, P_k)$ which is preserved by (4.11). Then Theorem 1.5 is an immediate consequence of the definition of the operator $\mathfrak{m}_k$ and Theorem 4.11. (We note that there may be a fixed point of (4.11). But this does not cause any problem as far as we work with *multi*-sections and study virtual fundamental chain over $\mathbb{Q}$.) □

Proof of Corollary 1.6. For $w : (D^2, \partial D^2) \to (M, L) = (M, \mathrm{Fix}\ \tau)$ we define its double $v : S^2 \to M$ by

$$v(z) = \begin{cases} w(z) & \text{for } z \in \mathbb{H} \\ \tau \circ w(\overline{z}) & \text{for } z \in \mathbb{C} \setminus \mathbb{H}, \end{cases}$$

where $(D^2, (-1, 1, \sqrt{-1}))$ is identified with the upper half plane $(\mathbb{H}, (0, 1, \infty))$ and $S^2 = \mathbb{C} \cup \{\infty\}$. Then it is easy to see that $c_1(TM)[v] = \mu_L([w])$. (See Example 4.7 (1).) Then the assumption (i) implies that $\mu_L \equiv 0 \mod 4$. Next we note the following general lemma.

Lemma 6.6. *Let L be an oriented Lagrangian submanifold of M. Then the composition*

$$\pi_2(M) \longrightarrow \pi_2(M, L) \xrightarrow{\mu_L} \mathbb{Z}$$

is equal to $2c_1(TM)[\alpha]$ for $[\alpha] \in \pi_2(M)$.

The proof is easy and so it is omitted. Then by this lemma the assumption (ii) also implies that the Maslov index of L modulo 4 is trivial. Therefore in either case of (i) and (ii), Theorem 1.5 implies $\mathfrak{m}_{0,\tau_*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$. On the other hand we have

$$\mathfrak{m}_0(1) = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{0,\beta}(1) T^{\omega(\beta)} e^{\mu(\beta)/2}$$

by Definition 6.1 and (6.3) which is also the same as

$$\sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{0,\tau_*\beta}(1) T^{\omega(\tau_*\beta)} e^{\mu(\tau_*\beta)/2}$$

because $\tau_*^2 = id$ and $\tau_* : \pi_2(M,L) \to \pi_2(M,L)$ is a one-one correspondence. Therefore since $\omega(\beta) = \omega(\tau_*\beta)$ and $\mu(\beta) = \mu(\tau_*\beta)$, we can rewrite $\mathfrak{m}_0(1)$ into

$$\mathfrak{m}_0(1) = \frac{1}{2} \sum_{\beta} \left(\mathfrak{m}_{0,\beta}(1) + \mathfrak{m}_{0,\tau_*\beta}(1)\right) T^{\omega(\beta)} e^{\mu(\beta)/2}$$

which becomes 0 by the above parity consideration. Hence L is unobstructed. Actually, we find that 0 is a bounding cochain; $0 \in \mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov})$. Furthermore, (1.2) implies

$$\mathfrak{m}_{2,\beta}(P_1, P_2) = (-1)^{1+\deg' P_1 \deg' P_2} \mathfrak{m}_{2,\tau_*\beta}(P_2, P_1). \tag{6.9}$$

We denote

$$P_1 \cup_Q P_2 := (-1)^{\deg P_1(\deg P_2 + 1)} \sum_{\beta} \mathfrak{m}_{2,\beta}(P_1, P_2) T^{\omega(\beta)} e^{\mu(\beta)/2}. \tag{6.10}$$

Then a simple calculation shows that (6.9) gives rise to

$$P_1 \cup_Q P_2 = (-1)^{\deg P_1 \deg P_2} P_2 \cup_Q P_1.$$

Hence $\cup_Q$ is graded commutative. □

Proof of Corollary 1.8. Let L be as in Corollary 1.8. By Corollary 1.6, L is unobstructed. Since $L = \mathrm{Fix}\ \tau$, we find that $c_1(TM)|\pi_2(M) = 0$ implies $\mu_L = 0$. Then Theorem E and Theorem 6.1.9 in [FOOO3] show that the Floer cohomology of L over $\Lambda^{\mathbb{Q}}_{nov}$ does not vanish for any $b \in \mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,nov})$:

$$HF((L,b); \Lambda^{\mathbb{Q}}_{nov}) \neq 0.$$

(Note that Theorem E holds not only over $\Lambda^{\mathbb{Q}}_{0,nov}$ but also over $\Lambda^{\mathbb{Q}}_{nov}$. See Theorem 6.1.9.) By extending the isomorphism (6.7) to $\Lambda^{\mathbb{Q}}_{nov}$ coefficients (by taking the tensor product with $\Lambda^{\mathbb{Q}}_{nov}$ over $\Lambda^{\mathbb{Q}}_{0,nov}$), we also have $HF((L,b),(L,b); \Lambda^{\mathbb{Q}}_{nov}) \neq 0$. Therefore by (6.8) we obtain

$$HF((\psi(L), \psi_* b), (L,b); \Lambda^{\mathbb{Q}}_{nov}) \neq 0$$

which implies $\psi(L) \cap L \neq \emptyset$. □

Proof of Theorem 1.9. Let (N, ω) be a symplectic manifold, $M = N \times N$, and $\omega_M = -p_1^*\omega_N + p_2^*\omega_N$. Denote by $p_i : N \times N \to N$ $(i = 1, 2)$ the i-th projection. We consider an anti-symplectic involution $\tau : M \to M$ defined by $\tau(x, y) = (y, x)$. Then $L = \mathrm{Fix}\,\tau \cong N$. Let J_N be a compatible almost structure on N, and $J_M = -J_N \otimes 1 + 1 \otimes J_N$. The almost complex structure J_M is compatible with ω_M. Note that $w_2(T(N \times N)) = p_1^* w_2(TN) + p_2^* w_2(TN)$.

If N is spin, then $L = \mathrm{Fix}\,\tau \cong N$ is τ-relatively spin by Example 3.12 and $c_1(T(N\times N)) \equiv w_2(T(N\times N)) \equiv 0 \mod 2$. Since $\pi_1(L) \to \pi_1(N\times N)$ is injective, Corollary 1.6 shows that L is unobstructed and $\mathfrak{m}_2$ defines a graded commutative product structure $\cup_Q$ by (6.10).

Suppose that N is not spin. We take a relative spin structure (V,σ) on $L = \mathrm{Fix}\,\tau \cong N$ such that $V = p_1^*(TN)$ and σ is the following spin structure on $(TL \oplus V)|_L \cong (TL\oplus TL)|_L$. Since the composition of the diagonal embedding $SO(n) \to SO(n)\times SO(n)$ and the inclusion $SO(n)\times SO(n) \to SO(2n)$ admits a unique lifting $SO(n) \to Spin(2n)$, we can equip the bundle $TL\oplus TL$ with a canonical spin structure. It determines the spin structure σ on $(TL\oplus V)|_L$. (In this case, we have $st = p_1^* w_2(TN)$.) Then clearly we find that $\tau^*V = p_2^*TN$. Note that p_1^*TN and p_2^*TN are canonically isomorphic to TL by the differentials of the projections p_1 and p_2, respectively. On the other hand, since $(TL \oplus \tau^*V)|_L \cong (TL\oplus TL)|_L \cong (TL\oplus V)|_L$, the spin structure σ is preserved by τ. Therefore the difference of the conjugacy classes of two relative spin structures $[(V,\sigma)]$ and $\tau^*[(V,\sigma)]$ is measured by $w_2(V\oplus\tau^*V) = w_2(p_1^*TN \oplus p_2^*TN)$. Using the canonical spin structure on $TL\oplus TL$ mentioned above, we can give a trivialization of $V\oplus\tau^*V$ over the 2-skeleton of L. Hence $w_2(V\oplus\tau^*V)$ is regarded as a class in $H^2(N\times N, L;\mathbb{Z}_2)$. Since $w_2(p_1^*TN \oplus p_2^*TN) = w_2(T(N\times N)) \equiv c_1(T(N\times N)) \mod 2$ and $\pi_2(N\times N) \to \pi_2(N\times N, L)$ is surjective, Lemma 6.6 shows that it is equal to $\mu_L/2$. Hence by Proposition 3.10 we obtain the following:

Lemma 6.7. *In the above situation, the identity map*

$$\mathcal{M}(J;\beta)^{[(V,\sigma)]} \longrightarrow \mathcal{M}(J;\beta)^{\tau^*[(V,\sigma)]}$$

is orientation preserving if and only if $\mu_L(\beta)/2$ is even.

Combining Theorem 1.4, we find that the composition

$$\begin{aligned}\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_1,\dots,P_k)^{[(V,\sigma)]} &\longrightarrow \mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]}\\ &\longrightarrow \mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]}\end{aligned}$$

is orientation preserving if and only if

$$k+1+\sum_{1\le i<j\le k} \deg' P_i \deg' P_j$$

is even. It follows that $\mathfrak{m}_{0,\tau_*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$ and hence we find that $L = \mathrm{Fix}\,\tau \cong N$ is unobstructred. This finishes the proof of the assertion (1). Moreover, we also find that $\mathfrak{m}_2$ satisfies Theorem 1.9 (6.9), which induces the graded commutative product $\cup_Q$ as well for the non-spin case.

Now we prove the isomorphism in Theorem 1.9 (2). Let $v : S^2 \to N$ be a J_N-holomorphic map. We fix 3 marked points $0,1,\infty \in S^2 = \mathbb{C}\cup\{\infty\}$. Then we consider the upper half plane $\mathbb{H}\cup\{\infty\} \subset \mathbb{C}\cup\{\infty\}$ and define a map $I(v) : \mathbb{H} \to M$ by

$$I(v)(z) = (v(\overline{z}), v(z)).$$

Identifying $(\mathbb{H},(0,1,\infty))$ with $(D^2,(-1,1,\sqrt{-1}))$ where $(-1,1,\sqrt{-1}) \in \partial D^2$, we obtain a map from $(D^2,\partial D^2)$ to (M,L) which we also denote by $I(v)$. One can easily check the converse: For any given J_M-holomorphic map $w : (D^2,\partial D^2) \cong$

$(\mathbb{H}, \mathbb{R} \cup \{\infty\}) \to (M, L) = (N \times N, \Delta_N)$, the assignment

$$v(z) = \begin{cases} p_2 \circ w(z) & \text{for } z \in \mathbb{H} \\ p_1 \circ w(\overline{z}) & \text{for } z \in \mathbb{C} \setminus \mathbb{H} \end{cases}$$

defines a J_N-holomorphic sphere on N. Therefore the map $v \mapsto I(v)$ gives an isomorphism between the moduli spaces of J_N-holomorphic spheres and J_M-holomorphic discs with boundary in N. We can easily check that this map is induced by the isomorphism of Kuranishi structures.

We remark however that this construction works only at the interior of the moduli spaces of pseudo-holomorphic discs and of pseudo-holomorphic spheres, that is the moduli spaces of those without bubble. To study the relationship between compactifications of them we need some extra argument, which will be explained later in this section.

We next compare the orientations on these moduli spaces. The moduli spaces of holomorphic spheres have canonical orientation, see, e.g., section 16 in [FO]. In Chapter 8 [FOOO3], we proved that a relative spin structure determines a system of orientations on the moduli spaces of bordered stable maps of genus 0. We briefly review a crutial step for comparing orientations in our setting.

Let $w : (D^2, \partial D^2) \to (M, L)$ be a J_M-holomorphic map. Denote by ℓ the restriction of w to ∂D^2. Consider the Dolbeault operator

$$\overline{\partial}_{(w^*TM, \ell^*TL)} : W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to L^p(D^2; w^*TM \otimes \Lambda^{0,1}_{D^2})$$

with $p > 2$. We deform this operator to an operator on the union Σ of D^2 and $\mathbb{C}P^1$ with the origin O of D^2 and the "south pole" S of $\mathbb{C}P^1$ identified. The spin structure σ on $TL \oplus V|_L$ gives a trivialization of $\ell^*(TL \oplus V|_L)$. Since w^*V is a vector bundle on the disc, it has a unique trivialization up to homotopy. Hence ℓ^*V inherits a trivialization, which is again unique up to homotopy. Using this trivialization, we can descend the vector bundle $E = w^*TM$ to E' on Σ. The index problem is reduced to the one for the Dolbeault operator on Σ. Namely, the restriction of the direct sum of the following two operators to the fiber product of the domains with respect to the evaluation maps at O and S. On D^2, we have the Dolbeault operator for the trivial vector bundle pair $(\underline{\mathbb{C}^n}, \underline{\mathbb{R}^n})$. On $\mathbb{C}P^1$, we have the Dolbeault operator for the vector bundle $E'|_{\mathbb{C}P^1}$. The former operator is surjective and its kernel is the space of constant sections in $\underline{\mathbb{R}^n}$. The latter has a natural orientation, since it is Dolbeault operator twisted by $E'|_{\mathbb{C}P^1}$ on a closed Riemann surface. Since the fiber product of kernels is taken on a complex vector space, the orientation of the index is determined by the orientations of the two operators.

Now we go back to our situation. Pick a 1-parameter family $\{\phi_t\}$ of dilations on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ such that $\lim_{t \to +\infty} \phi_t(z) = -\sqrt{-1}$ for $z \in \mathbb{C} \cup \{\infty\} \setminus \{\sqrt{-1}\}$. Here $\sqrt{-1}$ in the upper half plane and $-\sqrt{-1}$ in the lower half plane correspond to the north pole and the south pole of $\mathbb{C}P^1$, respectively. As $t \to +\infty$, the boundary of the second factor of the disc $I(v \circ \phi_t)$ contracts to the point $v(-\sqrt{-1})$ and its image exhausts the whole image of the sphere v, while the whole image of the first factor contracts to $v(-\sqrt{-1})$. Therefore as $t \to \infty$ the images of the map $z \mapsto I(v \circ \phi_t)(z)$ converge to the constant disc at $(v(-\sqrt{-1}), v(-\sqrt{-1}))$ with a sphere

$$z \in S^2 \mapsto (v(-\sqrt{-1}), v(z))$$

attached to the point. If we denote $w_t = I(v \circ \phi_t)$, it follows from our choice $V = p_1^* TN$ that the trivialization of $\ell_t^* V$, which is obtained by restricting the trivialization of $w_t^* V = (p_1 \circ I(v \circ \phi_t))^* TN$, coincides with the one induced by the frame of the fiber V at $v(-\sqrt{-1})$ for a sufficiently large t. Therefore considering the linearized index of the family w_t for a large t, it follows from the explanation given in the above paragraph that the map $v \mapsto I(v \circ \phi_t) = w_t$ induces an isomorphism

$$\det(\mathrm{Index} D\overline{\partial}_{J_N}(v)) \cong \det(\mathrm{Index} D\overline{\partial}_{J_M}(w_t))$$

as an oriented vector space. By flowing the orientation to $t = 0$ by the deformation ϕ_t, we have proven that the map $v \mapsto I(v)$ respects the orientations of the moduli spaces.

Now we compare the product $\cup_Q$ in (6.10) and the product on the quantum cohomology, presuming, for a while, that they can be calculated by the contribution from the interior of the moduli spaces only.

Define the equivalence relation $\sim$ on $\pi_2(N)$ by $\alpha \sim \alpha'$ if and only if $c_1(N)[\alpha] = c_1(N)[\alpha']$ and $\omega(\alpha) = \omega(\alpha')$. For $\beta = [w : (D^2, \partial D^2) \to (N \times N, \Delta_N)] \in \Pi(\Delta_N)$, we set $\widetilde{\beta} = [(p_2 \circ w) \# (p_1 \circ w) : D^2 \cup \overline{D}^2 \to N] \in \pi_2(N)/\sim$, where $\overline{D}^2$ is the unit disc with the complex structure reversed and $D^2 \cup \overline{D}^2$ is the union of discs glued along boundaries. This defines a homomorphism

$$\psi : \Pi(\Delta_N) \to \pi_2(N)/\sim . \tag{6.11}$$

For $\alpha \in \pi_2(N)$ let $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \alpha)$ be the moduli space of pseudo-holomorphic map $v : S^2 \to N$ of homotopy class α with three marked points, (without bubble). For $\rho \in \pi_2(N)/\sim$, we put

$$\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho) = \bigcup_{\alpha \in \rho} \mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \alpha).$$

For $\beta \in \Pi(\Delta_N)$, let $\mathcal{M}_3^{\mathrm{reg}}(J_{N\times N}; \beta)$ be the moduli space of pseudo-holomorphic map $u : (D^2, \partial D^2) \to (N \times N, \Delta_N)$ of class β with three boundary marked points (without disk or sphere bubble). We denote by $\mathcal{M}_3^{\mathrm{main,reg}}(\beta)$ its subset consisted of elements in the main component. We put

$$\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho) = \bigcup_{\psi(\beta)=\rho} \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \beta).$$

The above construction gives an isomorphism

$$I : \mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho) \to \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho) \tag{6.12}$$

of spaces with Kuranishi structure.

Denote by $*$ the cup product on the quantum cohomology $QH^*(N; \Lambda_{0,nov}^{\mathbb{Q}})$. By definition we have

$$\langle a_0, a_1 * a_2 \rangle = \sum_{\rho \in \pi_2(N)/\sim} (ev_0^* a_0 \cup ev_1^* a_1 \cup ev_2^* a_2)[\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho)] T^{\omega(\rho)} e^{c_1(N)[\rho]}.$$

Let us assume, for a moment that, the map $\mathfrak{m}_2$ is given by

$$\sum_{\beta; \psi(\beta)=\rho} \mathfrak{m}_{2,\beta}(P_1, P_2) T^{\omega(\beta)} e^{\mu(\beta)/2} = (\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho; P_1, P_2), ev_0) T^{\omega(\beta)} e^{\mu(\beta)/2}. \tag{6.13}$$

Here P_1 and P_2 are cycles, and

$$\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho; P_1, P_2) = \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho) \times_{N^2} (P_1 \times P_2).$$

(We also assume that the right hand side becomes a cycle.) This assumption is removed later in this subsection.

Taking a homological intersection number with another cycle P_0, we have

$$\begin{aligned}
&P_0 \cdot (\mathcal{M}_3^{\rm main,reg}(J_{N\times N};\rho;P_1,P_2),ev_0)\\
=&(-1)^{\epsilon}P_0\cdot\Big(\mathcal{M}_3^{\rm main,reg}(J_{N\times N};\rho)\times_{(ev_1,ev_2)}(P_1\times P_2),ev_0\Big)\\
=&(-1)^{\epsilon}(ev_0^*PD[P_0]\cup(ev_1,ev_2)^*PD[P_1\times P_2])[\mathcal{M}_3^{\rm main,reg}(J_{N\times N};\rho)],
\end{aligned}$$

where $PD[P_i]$, resp. $PD[P_j\times P_k]$ is the Poincaré dual of P_i in N, resp. $P_j\times P_k$ in $N\times N$. Since $\dim\Delta_N$ is even, $\epsilon=\deg P_1$, see Definition 3.15. Note that $ev_1^*PD[P_1]\cup ev_2^*PD[P_2]=(-1)^{\deg P_1\cdot\deg P_2}(ev_1,ev_2)^*PD[P_1\times P_2]$. By identifying $\mathcal{M}_3^{\rm sph,reg}(J_N;\rho)$ and $\mathcal{M}_3^{\rm main,reg}(J_{N\times N};\rho)$ as spaces with oriented Kuranishi structures, we find that

$$\langle PD[P_0],PD[P_1]*PD[P_2]\rangle=(-1)^{\deg P_1(\deg P_2+1)}\langle P_0,\mathfrak{m}_2(P_1,P_2)\rangle.$$

To complete the proof of Theorem 1.9, we need to remove the assumption (6.13). We study how our identification of the moduli spaces of pseudo-holomorphic discs (attached to the diagonal Δ_N) and of pseudo-holomorphic spheres (in N) extends to their compactifications for this purpose. To study this point, we define the isomorphism Theorem 1.9 (2) as $\Lambda_{0,nov}$ *modules* more explicitly.

As discussed in the introduction, this isomorphism follows from the degeneration at E_2-level of the spectral sequence of Theorem D [FOOO3]. The proof of this degeneration is based on the fact that the image of the differential is contained in the Poincaré dual to the kernel of the inclusion induced homomorphism $H(\Delta_N;\Lambda_{0,nov})\to H(N\times N;\Lambda_{0,nov})$, which is actually injective in our case. This fact (Theorem D (D.3) [FOOO3]) is proved by using the operator $\mathfrak{p}$ introduced in [FOOO3] Section 3.8. Therefore to describe this isomorphism we recall a part of the construction of this operator below.

Let $\beta\in\pi_2(N\times N,\Delta_N)$. We consider $\mathcal{M}_{1;1}(J_{N\times N};\beta)$, the moduli space of bordered stable maps of genus zero with one interior and one boundary marked point in homotopy class β. Let $ev_0:\mathcal{M}_{1;1}(J_{N\times N};\beta)\to\Delta_N$ be the evaluation map at the boundary marked point and $ev_{\rm int}:\mathcal{M}_{1;1}(J_{N\times N};\beta)\to N\times N$ be the evaluation map at the interior marked point. Let (P,f) be a smooth singular chain in Δ_N. We put

$$\mathcal{M}_{1;1}(J_{N\times N};\beta;P)=\mathcal{M}_{1;1}(J_{N\times N};\beta)\,{}_{ev_0}\times_f P.$$

It has a Kuranishi structure. We take its multisection $\mathfrak{s}$ and a triangulation of its zero set $\mathcal{M}_{1;1}(J_{N\times N};\beta;P)^{\mathfrak{s}}$. Then $(\mathcal{M}_{1;1}(J_{N\times N};\beta;P)^{\mathfrak{s}},ev_{\rm int})$ is a singular chain in $N\times N$, which is by definition $\mathfrak{p}_{1,\beta}(P)$. (See [FOOO3] Definition 3.8.23.) In our situation, where $\mathfrak{m}_0(1)=0$ in the chain level, we have:

Lemma 6.8. *We identify N with Δ_N. Then for any singular chain $P\subset N$ we have*

$$\delta_{N\times N}(\mathfrak{p}_{1,\beta}(P))+\mathfrak{p}_{1,\beta}(\mathfrak{m}_{1,0}(P))+\mathfrak{p}_{1,0}(\mathfrak{m}_{1,\beta}(P))+\sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}}\mathfrak{p}_{1,\beta_1}(\mathfrak{m}_{1,\beta_2}(P))=0.$$

Lemma 6.8 is a particular case of [FOOO3] Theorem 3.8.9 (3.8.10.2). We also remark $\mathfrak{p}_{1,0}(P)=P$. ([FOOO3] (3.8.10.1).)

Remark 6.9. In case $\beta = 0$, where $\overline{\mathfrak{p}}_{1,\beta} = \overline{\mathfrak{p}}_{1,0}$ is the identity map, the second and the third term coincides. In that case we do not have both but have only one term.

We remark that even in the case when P is a singular cycle $\mathfrak{m}_1(P)$ may not be zero. In other words the identity map

$$(C(\Delta_N;\Lambda_{0,nov}),\partial) \to (C(\Delta_N;\Lambda_{0,nov}),\mathfrak{m}_1) \tag{6.14}$$

is *not* a chain map. We use the operator

$$\mathfrak{p}_{1,\beta} : C(\Delta_N;\Lambda_{0,nov}) \to C(N \times N;\Lambda_{0,nov})$$

to modify the identity map to obtain a chain map (6.14). Let $p_2 : N \times N \to N$ be the projection to the second factor. We define

$$\overline{\mathfrak{p}}_{1,\beta} = p_{2*} \circ \mathfrak{p}_{1,\beta}.$$

Then by applying p_{2*} to the equation in Lemma 6.8, we obtain

$$-\mathfrak{m}_{1,0}(\overline{\mathfrak{p}}_{1,\beta}(P)) + \mathfrak{m}_{1,\beta}(P) + \overline{\mathfrak{p}}_{1,\beta}(\mathfrak{m}_{1,0}(P)) + \sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \overline{\mathfrak{p}}_{1,\beta_1}(\mathfrak{m}_{1,\beta_2}(P)) = 0. \tag{6.15}$$

Remark 6.10. The sign in Formula (6.15) follows from Lemma 6.8 except one for the first term. The sign of the first term is related to the choice of the sign of the boundary operator $\delta_{N\times N}$. It was not specified in [FOOO3], since it was not necessary there. We can determine the sign of the first term by considering the case when $\beta = 0$. In fact in that case the last term is zero and so the first term must cancel with the second (= third) term.

Definition 6.11. For each given singular chain P in N, we put

$$P(\beta) = \sum_{k=1}^{\infty} \sum_{\substack{\beta_1+\cdots+\beta_k=\beta\\ \beta_i\neq 0}} (-1)^k \left(\overline{\mathfrak{p}}_{1,\beta_1} \circ \cdots \circ \overline{\mathfrak{p}}_{1,\beta_k}\right)(P)$$

regarding P as a chain in Δ_N. Then we define a chain $\mathfrak{I}(P) \in C(N;\Lambda_{0,nov})$ by

$$\mathfrak{I}(P) = P + \sum_{\beta\neq 0} P(\beta) T^{\omega(\beta)} e^{\mu(\beta)/2}$$

Lemma 6.12.

$$\mathfrak{I} : (C(\Delta_N;\Lambda_{0,nov}),\mathfrak{m}_{1,0}) \to (C(\Delta_N;\Lambda_{0,nov}),\mathfrak{m}_1)$$

is a chain homotopy equivalence.

Proof. We can use (6.15) to show that $\mathfrak{I}$ is a chain map as follows. We prove that $\mathfrak{m}_1 \circ \mathfrak{I} - \mathfrak{I} \circ \mathfrak{m}_{1,0} \equiv 0 \mod T^{\omega(\beta)}$ by induction on $\omega(\beta)$. We assume that it holds modulo $T^{\omega(\beta)}$ and will study the terms of order $T^{\omega(\beta)}$. Those terms are sum of

$$\mathfrak{m}_{1,0}(P(\beta)) + \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_i\neq 0}} \mathfrak{m}_{1,\beta_1}(P(\beta_2)), \tag{6.16}$$

for given $\omega(\beta)$'s. By definition (6.16) is:

$$\begin{aligned} &\sum_{k=1,2,\dots} \sum_{\substack{\beta_1+\cdots+\beta_k=\beta\\ \beta_i\neq 0}} (-1)^k \left(\mathfrak{m}_{1,0} \circ \overline{\mathfrak{p}}_{1,\beta_1} \circ \cdots \circ \overline{\mathfrak{p}}_{1,\beta_k}\right)(P) \\ &+ \sum_{k=1,2,\dots} \sum_{\substack{\beta_1+\cdots+\beta_k=\beta\\ \beta_i\neq 0}} (-1)^{k-1} \left(\mathfrak{m}_{1,\beta_1} \circ \overline{\mathfrak{p}}_{1,\beta_2} \circ \cdots \circ \overline{\mathfrak{p}}_{1,\beta_k}\right)(P). \end{aligned} \tag{6.17}$$

Using (6.15) we can show that (6.17) is equal to

$$\begin{aligned}&\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k\left(\overline{\mathfrak p}_{1,\beta_1}\circ\mathfrak m_{1,0}\circ\overline{\mathfrak p}_{1,\beta_2}\circ\dots\circ\overline{\mathfrak p}_{1,\beta_k}\right)(P)\\ &+\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k\left(\overline{\mathfrak p}_{1,\beta_1}\circ\mathfrak m_{1,\beta_2}\circ\overline{\mathfrak p}_{1,\beta_3}\circ\dots\circ\overline{\mathfrak p}_{1,\beta_k}\right)(P).\end{aligned}\tag{6.18}$$

(6.18) vanishes by induction hypothesis.

On the other hand $\mathfrak I$ is identity modulo $\Lambda^+_{0,nov}$. The lemma follows. □

Thus we obtain an isomorphism

$$\mathfrak I_\# : H(N;\Lambda_{0,nov}) \cong HF(\Delta_N,\Delta_N;\Lambda_{0,nov}).$$

In order to complete the proof of Theorem 1.9, we need to prove

$$\pm\langle\mathfrak m_2(\mathfrak I(P_1),\mathfrak I(P_2)),\mathfrak I(P_0)\rangle = \langle PD[P_1]*PD[P_2],PD[P_0]\rangle.\tag{6.19}$$

(We already discussed the sign in detail. So we omit the discussion about it later on in this subsection.) We will prove (6.19) in the rest of this subsection.

We consider a class $\rho\in\pi_2(N)/\sim$. Let

$$\mathcal M_3^{\rm sph}(J_N;\rho)=\bigcup_{\alpha\in\rho}\mathcal M_3^{\rm sph}(J_N;\alpha)$$

where $\mathcal M_3^{\rm sph}(J_N;\alpha)$ be the moduli space of stable map of genus 0 with 3 marked points and of homotopy class α. Let $((\Sigma,(z_0,z_1,z_2)),v)$ be a representative of its element. We decompose $\Sigma=\bigcup\Sigma_a$ to irreducible components.

Definition 6.13. (1) Let $\Sigma_0\subset\Sigma$ be the minimal connected union of irreducible components containing three marked points z_0,z_1,z_2.

(2) An irreducible component Σ_a is said to be *Type I* if it is contained in Σ_0. Otherwise it is said to be *Type II*.

(3) Let Σ_a be an irreducible component of Type I. Let k_a be the number of its singular points in Σ_a which do not intersect irreducible components of $\Sigma\setminus\Sigma_0$. Let k'_a be the number of marked points on Σ_a. It is easy to see that $k_a+k'_a$ is either 2 or 3. We say that Σ_a is *Type I-1* if $k_a+k'_a=3$ and *Type I-2* if $k_a+k'_a=2$. We say those $k_a+k'_a$ points, *the interiror special points*.

We remark that there exists exactly one irreducible component of Type I-1.

Our next task is to describe the double of a bordered stable map of genus zero with three boundary marked points. We do so by describing the image of the boundary circles in the double. For this purpose we define the notion of admissible system of circles below.

Let Σ_a be an irreducible component of Σ where $((\Sigma,(z_0,z_1,z_2)),v)$ is an element of $\mathcal M_3^{\rm sph}(J_N;\alpha)$. An oriented small circle C_a in $\Sigma_a\cong S^2$ is said to be a circle here for simplicity. A domain which bounds C_a is a disc in S^2 whose boundary (together with orientation) is C_a. By an abuse of notation, we also include the case when C_a is a point. In that case its orientation is, by definition, a choice between a point ($=C_a$) and the whole component of Σ_a itself, which we regard as the 'disc which bounds C_a'.

Now consider a component Σ_a of Type I-2. We say an interior special point of Σ_a is *inward* if it is contained in the connected component of the closure of $\Sigma_0\setminus\Sigma_a$

that contains the unique Type I-1 component. Otherwise it is called *outward*. (An inward interior special point must necessarily be a singular point.)

Definition 6.14. An admissible system of circles of Type I for $((\Sigma,(z_0,z_1,z_2)),v)$ is an assignment of a circle to each of the irreducible components Σ_a of Type I, such that the following holds:

(1) If Σ_a is Type I-1, C_a contains all the three interior special points.
(2) Let Σ_a is Type I-2. We require that C_a contains the outward interior special point and the domain bounding C_a contains the inward one.
(3) Denote by Σ_{a_0} the unique irreducible component of Type I-1 and let C be the maximal connected union of C_a's containing C_{a_0}. If C contains all z_i, we require the orientation of C to respect the cyclic order of (z_0,z_1,z_2). If some z_i is not on C, we instead consider the following point z_i' on C described below and ask the orientation of C respects the cyclic order of (z_0',z_1',z_2'): There exists a unique irreducible component Σ_a such that C is contained in a connected component of the closure of $\Sigma_0\setminus\Sigma_a$, z_i is contained in the other connected component, and that C intersects Σ_a (at the inward interior special point of Σ_a). Then the point z_i' is this inward interior special point of Σ_a.

Example 6.15. (1) Let us consider the admissible system of circles as in Figure 1 below.

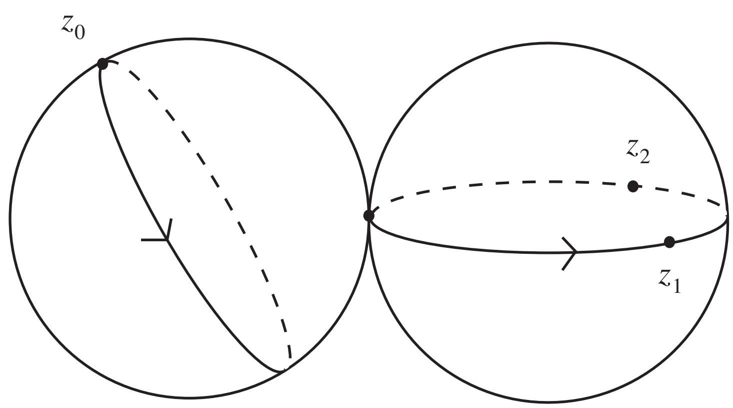

Figure 1

This is the double of the following configuration:

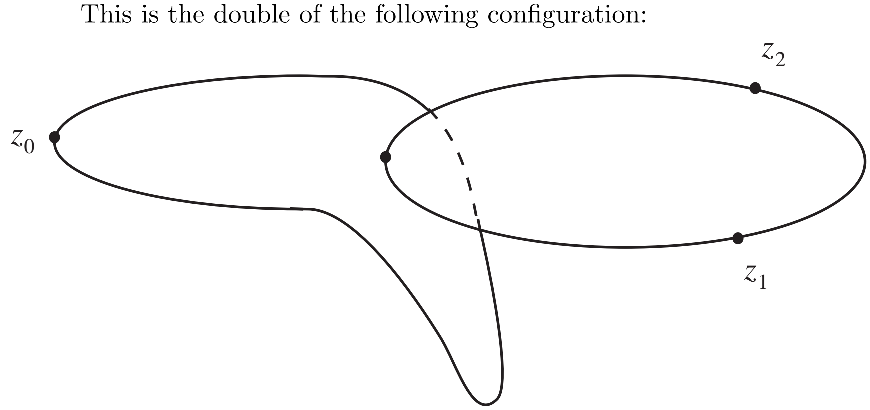

Figure 2

The moduli space of such configurations is identified with the moduli space that is used to define

$$\langle \mathfrak{m}_{2,\beta_2}(P_1, P_2), \overline{\mathfrak{p}}_{1,\beta_1}(P_0)\rangle. \tag{6.20}$$

(2) The case of the configuration of the following Figure 3 is also contained in the moduli space that is used to define (6.20). (This is the case when the circle C_a is associated to the component in the middle consists of one point.) Figure 3 corresponds to the case when interior marked point of $\mathcal{M}_{1;1}(J_{N\times N}; \beta_1)$ (which appears in the definition of $\overline{\mathfrak{p}}_{1,\beta_1}$) is on the sphere bubble. (See Figure 4.)

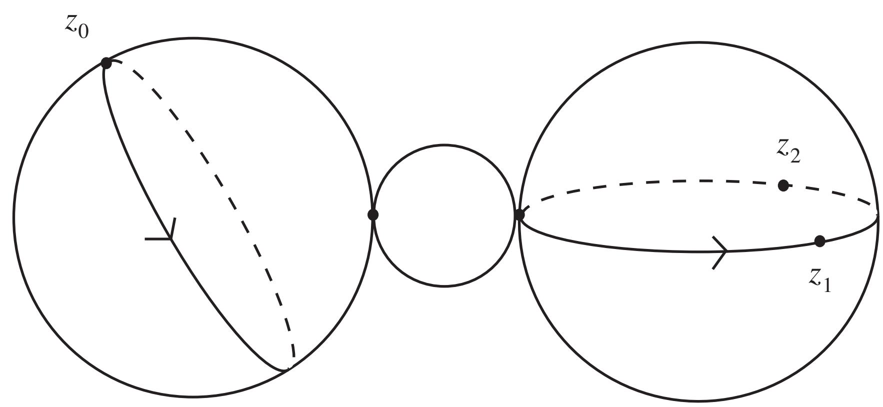

Figure 3

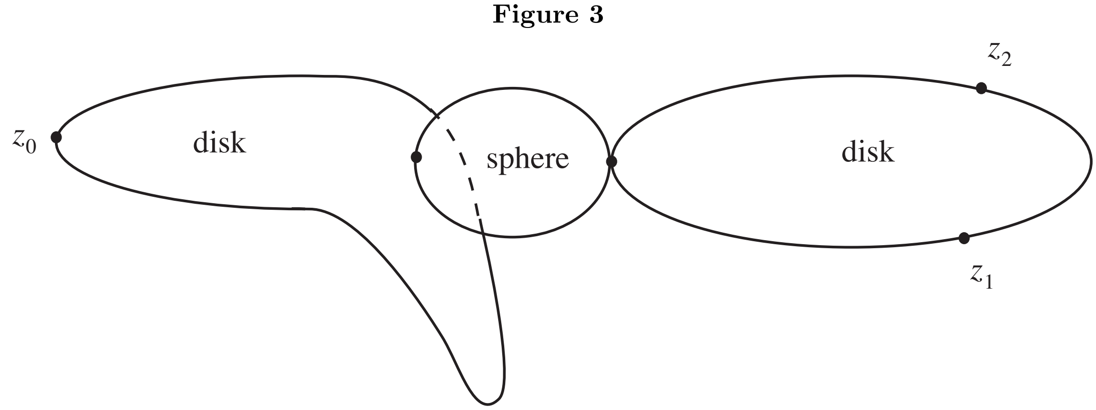

Figure 4

We next discuss the admissible system of circles on the irreducible components of Type II. A *connected component of Type II* of Σ is by definition the closure of a connected component of $\Sigma \setminus \Sigma_0$. Each connected component of Type II intersects Σ_0 at one point. We call this point the *root* of our connected component of Type II.

We denote by Σ_ρ a connected component of Type II and express it as

$$\Sigma_\rho = \bigcup_{a\in I_\rho} \Sigma_a.$$

Then we consider a Type II irreducible component Σ_a contained in a Σ_ρ. If Σ_a does not contain the root of Σ_ρ, we consider the connected component of the closure of

$\Sigma_\rho \setminus \Sigma_a$ that contains the root of Σ_ρ. Then, there is a unique singular point of Σ_a contained therein. We call this singular point the *root* of Σ_a. Note that if Σ_a contains the root of Σ_ρ, it is, by definition, the root of Σ_a.

Definition 6.16. Let an admissible system of circles of Type I on Σ be given. We define an admissible system of circles of Type II on Σ_ρ to be a union

$$C_\rho = \bigcup_{a \in I_\rho} C_a$$

in which C_a is either a circle or an empty set and which we require to satisfy the following:

(1) If the root of Σ_ρ is not contained in our system of circles of Type I, then all of C_a are empty set.
(2) If C_ρ is nonempty, then it is connected and contains the root of Σ_ρ.
(3) Let Σ_a be a Type II irreducible component contained in the Type II connected component Σ_ρ. Let Σ_b be the irreducible component of Σ that contains the root of Σ_a and assume $a \neq b$. If the root of Σ_a is contained in C_b, we require C_a to be nonempty.

Definition 6.17. An admissible system of circles on Σ is, by definition, an admissible system of circles of Type I together with that of Type II on each connected component of Type II.

We denote by $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ the moduli space consisting of pairs of an element $((\Sigma,\vec z),u) \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$ and admissible system of circles on Σ. We put

$$\begin{aligned}\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0) &= \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})_{(ev_1,ev_2,ev_0)} \times (P_1 \times P_2 \times P_0)\\ \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;P_1,P_2,P_0) &= \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)_{(ev_1,ev_2,ev_0)} \times (P_1 \times P_2 \times P_0).\end{aligned}$$

For $\rho \in \pi_2(N)/\sim$ we define $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ and $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C})$ in an obvious way. We denote by $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\overline{\mathcal{C}})$ the moduli space consisting of the pairs of an element $((\Sigma,\vec z),u) \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$ and an admissible system of circles of *Type I* on Σ. The notations $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\overline{\mathcal{C}};P_1,P_2,P_0)$ etc. are defined in the same way.

We can provide a natural Kuranishi structure on each of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$, $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ and $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\overline{\mathcal{C}})$ in a straightforward way, and hence also on their fiber products with singular chains P_i's.

The following lemma is a well-known fact on the moduli space of pseudo-holomorphic spheres which is used in the definition of quantum cup product [FO].

Lemma 6.18. *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ carries a Kuranishi structure and a multisection $\mathfrak{s}_0$ such that*

$$\sum_\rho \# \left(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)\right)^{\mathfrak{s}_0} T^{\omega(\alpha)} e^{c_1(N)[\alpha]} = \langle PD[P_1] * PD[P_2], PD[P_0]\rangle$$

where the sum is taken over ρ for which the virtual dimension of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ is zero.

Now we consider the moduli space used to define the left hand side of (6.19). Let $\vec\beta_i = \beta_{i,1},\dots,\beta_{i_{k_i}} \in \pi_2(N\times N,\Delta_N)$. We define

$$\begin{aligned}\mathcal{M}_{1;1}(J_{N\times N};\vec\beta_i;P_i) =& \mathcal{M}_{1;1}(J_{N\times N};\beta_{i_1};P_i) \,{}_{p_2\circ ev_{\mathrm{int}}}\times_{ev_0} \mathcal{M}_{1;1}(J_{N\times N};\beta_{i_2})\\ &{}_{p_2\circ ev_{\mathrm{int}}}\times_{ev_0} \cdots {}_{p_2\circ ev_{\mathrm{int}}}\times_{ev_0} \mathcal{M}_{1;1}(J_{N\times N};\beta_{i_{k_i}})\end{aligned} \tag{6.21}$$

and

$$\begin{aligned}&\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)\\&=\mathcal{M}_3^{\text{main}}(J_{N\times N};\beta')\ {}_{(ev_1,ev_2,ev_0)}\times_{(p_2\circ ev_{\text{int}})^3}\prod_{i=1,2,0}\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta}_i;P_i).\end{aligned}\tag{6.22}$$

The following is immediate from definition.

Lemma 6.19. *There exist a Kuranishi structure and a multisection $\mathfrak{s}_1$ with the following properties. We denote by n_β the sum of*

$$\#\left(\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)\right)^{\mathfrak{s}_1}$$

over $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ whose total sum is β. Then we have

$$\langle\mathfrak{m}_2(\mathfrak{I}(P_1),\mathfrak{I}(P_2)),\mathfrak{I}(P_0)\rangle=\sum_\beta n_\beta T^{\omega(\beta)}e^{\mu(\beta)/2}.\tag{6.23}$$

Consider the union of $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ over $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ such that the total sum of $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ is β whose double belongs to class $\rho\in\pi_2(N)/\sim$. We denote the union by $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$. We then define a map

$$\mathfrak{I}:\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)\to\mathcal{M}_3^{\text{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)\tag{6.24}$$

as follows. Let $(p_i,(S_{i,j},(z_{i,j;0},z_{i,j;\text{int}}),u_{i,j})_{j=1}^{k_i})$ be an element of $\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta}_i;P_i)$. Here $p_i\in P_i$, $(S_{i,j},(z_{i,j;0},z_{i,j;\text{int}}),u_{i,j})\in\mathcal{M}_{1,1}(J_{N\times N};\beta_{i,j})$ such that

$$f(p_i)=u_{i,1}(z_{i,1;0}),\ u_{i,1}(z_{i,1;\text{int}})=u_{i,2}(z_{i,2;0}),\ \ldots,\ u_{i,k_i-1}(z_{i,k_i-1;\text{int}})=u_{i,k_i}(z_{i,2;k_i}),$$

where P_i is $(|P_i|,f)$, $|P_i|$ is a simplex, and $f:|P_i|\to N$ is a smooth map.

Writing $u_{i,j}=(u^+_{i,j},u^-_{i,j})$, we obtain a map $\widehat{u}_{i,j}:\Sigma_{i,j}\to N$ from a sphere $\Sigma_{i,j}$ the double of $S_{i,j}$. (See the definition of the map I, (6.12).)

We denote by $C_{i,j}\subset\Sigma_{i,j}$ the circle along which we glued two copies of $S_{i,j}$. Then $\widehat{u}_{i,j}$ is defined by gluing $u^+_{i,j}$ and $u^-_{i,j}\circ c$ along $C_{i,j}$ in $\Sigma_{i,j}$ where $c:\Sigma_{i,j}\to\Sigma_{i,j}$ is the conjugation with $C_{i,j}$ as its fixed point set.

We glue $(\Sigma_{i,j},u_{i,j})$ and $(\Sigma_{i,j+1},u_{i,j+1})$ at $z_{i,j;\text{int}}$ and $z_{i,j;0}$. Here we identify $z_{i,j;\text{int}}\in S_{i,j}$ as the corresponding point in $\Sigma_{i,j}$ such that it is in the disc bounding $C_{i,j}$. We thus obtain a configuration of tree of spheres and system of circles on it, for each $i=1,2,0$. We glued them with the double of an element $\mathcal{M}_3^{\text{main}}(J_{N\times N};\beta')$ in an obvious way. Thus we obtain the map (6.24). (In case some of the sphere component becomes unstable we need to shrink it. See the proof of Lemma 6.20 below.)

It is easy to see that $\mathfrak{I}$ is surjective.

Lemma 6.20. *The map $\mathfrak{I}$ is an isomorphism outside the set of codimension ≥ 2.*

Proof. We can easily check that the map $\mathfrak{I}$ fails to be an isomorphism only by the following reason. Let $((\Sigma^{\text{dis}},(z_1,z_2,z_0)),v)$ be an element of $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$ and let Σ_i^{dis} be one of its irreducible sphere component. Suppose that Σ_i^{dis} is unstable. (Namely we assume that it has one or two singular points.) Then its automorphism group $\text{Aut}(\Sigma_i^{\text{dis}})$ will have positive dimension by definition of stability. (We require the elements of $\text{Aut}(\Sigma_i^{\text{dis}})$ to fix the singular point.) By restricting v to Σ_i^{dis}, we obtain $v_i=(v_i^+,v_i^-)$ where $v_i^\pm:\Sigma_i^{\text{dis}}\to N$ are maps from the sphere domain Σ_i^{dis}. On the double (which represents $\mathfrak{I}((\Sigma^{\text{dis}},(z_1,z_2,z_0)),v)$) the domain

$\Sigma^{\rm dis}$ contains two sphere components Σ_i^+ and Σ_i^- on which the maps v_i^+ and v_i^- are defined respectively. We have two alternatives:

(1) If one of v_i^+ and v_i^- is a constant map, then this double itself is not a stable map. So we shrink the corresponding component Σ_i^+ or Σ_i^- to obtain a stable map. (This is actually a part of the construction used in the definition of $\mathfrak{I}$.)

(2) Suppose both v_i^+ and v_i^- are nonconstant and let $g \in \mathrm{Aut}(\Sigma_i^{\rm dis})$. Then the map $v_i^g = (v_i^+, v_i^- \circ g)$ defines an element of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ different from $v_i = (v_i^+, v_i^-)$ but is mapped to the same element under the map $\mathfrak{I}$.)

This phenomenon occurs only at the stratum of codimension ≥ 2 because it occurs only when there exists a sphere bubble. This finishes the proof. □

Lemma 6.21. *We consider the involution τ_* applied to one of the disc component S of the fiber product factors appearing in a stratum of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$, such that the double of S is Type II. Then the orientation of the circle $C = \partial S$ embedded in the domain Σ of the corresponding sphere component is inverted under the operation on $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ induced by τ_* under the map $\mathfrak{I}$.*

This is immediate from construction and so its proof is omitted. □

Lemma 6.22. *The moduli space $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ carries a Kuranishi structure which can be canonically pull-backed to a Kuranishi structure on the space $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$.*

Proof. This is also clear from construction except the following point: At the point of the moduli space $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ that corresponds to

(1) either a circle in Type II component Σ_a hits the singular point of Σ_a other than the root thereof,

(2) or a circle in Type I-2 component Σ_a hits the singular point other than its outward interior special point,

construction of a Kuranishi neighborhood of the point in $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ is not so trivial, because various different strata meet at the point. However at these points, we can examine the way how the corresponding strata are glued in $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ and glue the corresponding strata of $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ in the same way.

We like to note that the phenomenon spelled out in the proof of Lemma 6.20 concerns *sphere* bubbles of the elements of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$, while the phenomenon we concern here concerns *disc* bubbles. Therefore they do not interfere with each other. □

Corollary 6.23. *The Kuranishi structure and its multisection $\mathfrak{s}_1$ given in Lemma 6.19 can be chosen so that it pulls back from a Kuranishi structure and a multisection $\mathfrak{s}_2$ on $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$. In particular we have*

$$\sum_\beta n_\beta = \# \left(\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)\right)^{\mathfrak{s}_2}.$$

Here the sum is taken over the β whose double belongs to class $\rho \in \pi_2(N)/\sim$.

We now consider the forgetful maps:

$$\begin{aligned}
\mathfrak{F}_1 : \mathcal{M}_3^{\rm sph}(J_N;\rho;\mathcal{C};P_1,P_2,P_0) &\to \mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0), && (6.25)\\
\mathfrak{F}_2 : \mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0) &\to \mathcal{M}_3^{\rm sph}(J_N;\rho;P_1,P_2,P_0). && (6.26)
\end{aligned}$$

$\mathfrak{F}_1$ is defined by forgetting all the circles of Type II and $\mathfrak{F}_2$ is defined by forgetting all the circles of Type I. We recall that all three moduli spaces $\mathcal{M}_3^{\rm sph}(J_N;\rho;P_1,P_2,P_0)$, $\mathcal{M}_3^{\rm sph}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ and $\mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0)$ carry Kuranishi structures (see Definition 6.17). We have also used multisections on $\mathcal{M}_3^{\rm sph}(J_N;\rho;P_1,P_2,P_0)$ and on $\mathcal{M}_3^{\rm sph}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$, denoted by $\mathfrak{s}_0$ and $\mathfrak{s}_2$ respectively.

Let Z be a compact metrizable space and $\mathfrak{K}_0$, $\mathfrak{K}_1$ its Kuranishi structures with orientation. Let $\mathfrak{s}_0$ and $\mathfrak{s}_1$ be multisections of the Kuranishi structures $\mathfrak{K}_0$ and $\mathfrak{K}_1$, respectively. We say that $(Z,\mathfrak{K}_0,\mathfrak{s}_0)$ is homotopic to $(Z,\mathfrak{K}_1,\mathfrak{s}_1)$ if there exists an oriented Kuranishi structure on $Z\times[0,1]$ and its multisection $\mathfrak{s}$ which restricts to $(\mathfrak{K}_0,\mathfrak{s}_0)$ and $(\mathfrak{K}_1,\mathfrak{s}_1)$ at $Z\times\{0\}$ and $Z\times\{1\}$, respectively.

We observe that the fiber of the map $\mathfrak{F}_2$ is a product of discs. (At the stratum which has k Type I-2 irreducible components, the fiber is $(D^2)^k$.) In particular the fiber is contractible.

Lemma 6.24. *Let $(\mathfrak{K}_0,\mathfrak{s}_0)$ be a pair of Kuranishi structure and its multisection on $\mathcal{M}_3^{\rm sph}(J_N;\rho;P_1,P_2,P_0)$ and $(\mathfrak{K}_3,\mathfrak{s}_3)$ a pair on $\mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0)$. Then $(\mathfrak{K}_3,\mathfrak{s}_3)$ is homotopic to the pull-back $\mathfrak{F}_2^*(\mathfrak{K}_0,\mathfrak{s}_0)$. In particular we have*

$$\#\left(\mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0)\right)^{\mathfrak{s}_3} = \#\left(\mathcal{M}_3^{\rm sph}(J_N;\rho;P_1,P_2,P_0)\right)^{\mathfrak{s}_0}. \qquad (6.27)$$

Proof. We can easily pull-back $\mathfrak{s}_0$ by $\mathfrak{F}_2$. Then the existence of homotopy of Kuranishi structure and its multisection is a consequence of general theory of Kuranishi structure. To prove the equality (6.27) it suffices to see that the boundary of the moduli space $\mathcal{M}_3^{\rm sph}(J_N;\rho;\overline{\mathcal{C}};P_1,P_2,P_0)$ is empty. Actually there are the following two kinds of strata which could potentially contribute to the boundary:

(1) A Type I component splits into two.
(2) A Type I disk C_a meets an inward interior marked point.

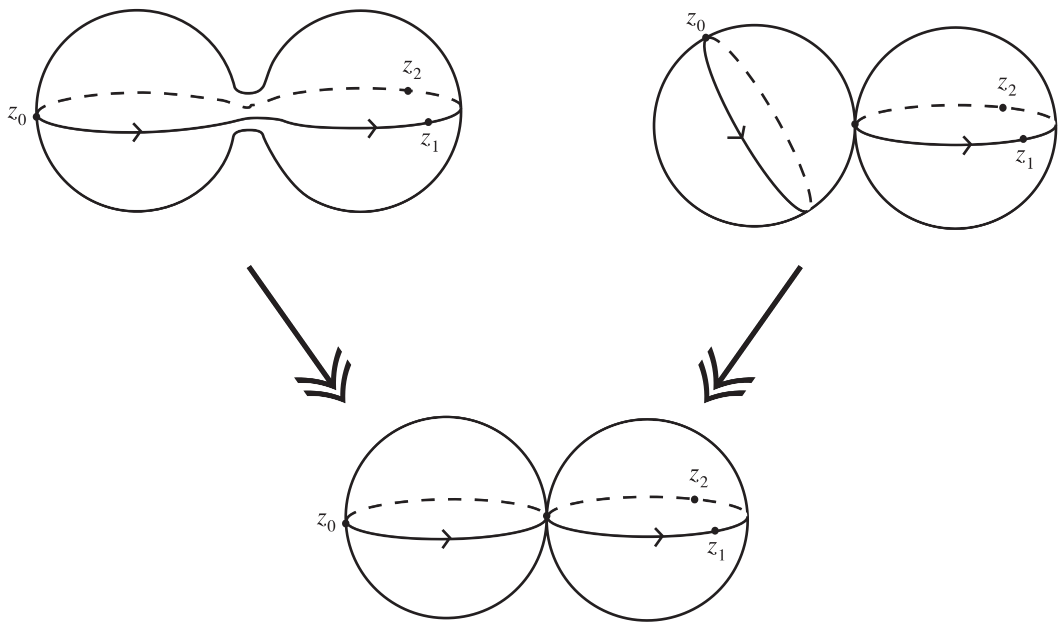

Figure 5

These two strata cancel each other. This is the geometric way to see the equality in Lemma 6.8. □

Lemma 6.25. *Let $(\mathfrak{K}_3, \mathfrak{s}_3)$ be a pair of Kuranishi structure and its multisection on $\mathcal{M}_3^{\rm sph}(J_N; \rho; \overline{\mathcal{C}}; P_1, P_2, P_0)$ and $(\mathfrak{K}_2, \mathfrak{s}_2)$ a pair on $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$. Then $(\mathfrak{K}_2, \mathfrak{s}_2)$ is homotopic to the pull-back $\mathfrak{F}_1^*(\mathfrak{K}_3, \mathfrak{s}_3)$.*

We may choose the homotopy so that it is invariant under the inversion of the orientation of the Type II circles. In particular we have

$$\# \left(\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)\right)^{\mathfrak{s}_2} = \# \left(\mathcal{M}_3^{\rm sph}(J_N; \rho; \overline{\mathcal{C}}; P_1, P_2, P_0)\right)^{\mathfrak{s}_3}.$$

Proof. The first half is again a consequence of general theory. We remark that there are several components of $\mathcal{M}_3^{\rm sph}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ which are of codimension 0 or 1 and contracted by $\mathfrak{F}_1$. All the contribution from those components cancel out by the involution, which invert the orientation of the circles of Type II. (This is a geometric way to see the vanishing of $\mathfrak{m}_0(1)$ in the chain level. We have already checked that it occurs *with sign*.) Hence the lemma. □

The proof of Theorem 1.9 is now complete. □

Proof of Corollary 1.10. Viewing N as a closed relatively spin Lagrangian submanifold of $(N \times N, -p_1^*\omega_N + p_2^*\omega_N)$, we can construct a filtered A_∞ structure on $H(N; \Lambda_{0,nov}^{\mathbb{Q}})$ which is homotopy equivalent to the filtered A_∞ algebra given by Theorem 6.2. This is a consequence of Theorem W [FOOO3]. See also Theorem A [FOOO3]. Then (1) and (2) follow from Theorem 1.9. The assertion (3) follows from Theorem X [FOOO3]. □

6.3. **Calculation of Floer cohomology of $\mathbb{R}P^{2n+1}$.** In this subsection, we apply the results proved in the previous sections to calculate Floer cohomology of real projective space of odd dimension. Since the case $\mathbb{R}P^1 \subset \mathbb{C}P^1$ is already

discussed in Subsection 3.7.6 [FOOO3], we consider $\mathbb{R}P^{2n+1}$ for $n > 0$. We note that $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ is monotone with minimal Maslov index $2n+2 > 2$ if $n > 0$. Therefore by [Oh1] and Section 2.4 [FOOO3] Floer cohomology over $\Lambda_{0,nov}^{\mathbb{Z}}$ is defined. In this case we do not need to use the notion of Kuranishi structure and the technique of the virtual fundamental chain. From the proof of Corollary 1.6, we can take 0 as a bounding cochain. Hereafter we omit the bounding cochain 0 from the notation. By [Oh2] and Theorem D in [FOOO3], we have a spectral sequence converging to the Floer cohomology. Strictly speaking, in [Oh2] the spectral sequence is constructed over $\mathbb{Z}_2$ coefficient. However, we can generalize his results to ones over $\Lambda_{0,nov}^{\mathbb{Z}}$ coefficient in a straightforward way, as long as we take the orientation problem, which is a new and crucial point of this calculation here, into account. Thus Oh's spectral sequence over $\Lambda_{0,nov}^{\mathbb{Z}}$ is enough for our calculation of this example. (See Chapter 8 and Chapter 6 in [FOOO2] for a spectral sequence over $\Lambda_{0,nov}^{\mathbb{Z}}$ in more general setting.)

We use a relative spin structure in Proposition 3.14 when n is even and a spin structure when n is odd. We already check that $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ has two inequivalent relative spin structures. The next theorem applies to both of them.

Theorem 6.26. *Let n be any positive integer. Then the spectral sequence calculating $HF(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n+1}; \Lambda_{0,nov}^{\mathbb{Z}})$ has unique nonzero differential*

$$d^{2n+1} : H^{2n+1}(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H^0(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}$$

which is multiplication by ± 2. In particular, we have

$$HF(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n+1}; \Lambda_{0,nov}^{\mathbb{Z}}) \cong (\Lambda_{0,nov}^{\mathbb{Z}}/2\Lambda_{0,nov}^{\mathbb{Z}})^{\oplus(n+1)}.$$

Remark 6.27. (1) Floer cohomology of $\mathbb{R}P^m$ over $\mathbb{Z}_2$ is calculated in [Oh1] and is isomorphic to the ordinary cohomology. This fact also follows from Theorem 34.16 in [FOOO2], which implies that Floer cohomology of $\mathbb{R}P^m$ over $\Lambda_{0,nov}^{\mathbb{Z}_2}$ is isomorphic to the ordinary cohomology over $\Lambda_{0,nov}^{\mathbb{Z}_2}$.

(2) Theorem 6.26 gives an example where Floer cohomology of the real point set is different from its ordinary cohomology. Therefore it is necessary to use $\mathbb{Z}_2$ coefficient to study the Arnold-Givental conjecture (see Chapter 8 [FOOO2]).

(3) In Subsection 3.6.3 [FOOO3], we introduced the notion of *weak unobstructedness* and *weak bounding cochains* using the homotopy unit of the filtered A_∞ algebra. We denote by $\mathcal{M}_{\rm weak}(L; \Lambda_{0,nov})$ the set of all weak bounding cochains. We also defined the potential function $\mathfrak{PO} : \mathcal{M}_{\rm weak}(L; \Lambda_{0,nov}) \to \Lambda_{0,nov}^{+(0)}$, where $\Lambda_{0,nov}^{+(0)}$ is the degree zero part of $\Lambda_{0,nov}^{+}$. Then the set of bounding cochains $\mathcal{M}(L; \Lambda_{0,nov})$ is characterized by $\mathcal{M}(L; \Lambda_{0,nov}) = \mathfrak{PO}^{-1}(0)$. As we see above, $\mathbb{R}P^{2n+1}$ is unobstructed. Thus $\mathfrak{PO}(b) = 0$ for any bounding cochain b. About the value of the potential function, we have the following problem:

Problem 6.28. Let L be a relatively spin Lagrangian submanifold of a symplectic manifold M. We assume that L is weakly unobstructed and that the Floer cohomology $HF((L,b),(L,b); \Lambda_{0,nov}^{F})$ deformed by $b \in \mathcal{M}_{\rm weak}(L)$ does not vanish for some field F. In this situation, the question is whether $\mathfrak{PO}(b)$ is an eigenvalue of the operation

$$c \mapsto c \cup_Q c_1(M) : QH(M; \Lambda_{0,nov}^{F}) \longrightarrow QH(M; \Lambda_{0,nov}^{F}).$$

Here $(QH(M; \Lambda_{0,nov}^{F}), \cup_Q)$ is the quantum cohomology ring of M over $\Lambda_{0,nov}^{F}$.

Such statement was made by M. Kontsevich in 2006 during a conference of homological mirror symmetry at Vienna. (According to some physicists this had been known to them before.) See also [A]. In the case that $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$, we have $\mathfrak{PO}(b) = 0$ for any bounding cochain b as we observed above. On the other hand, Theorem 6.26 shows that the Floer cohomology does not vanish for $F = \mathbb{Z}_2$, and the eigenvalue is zero in the field $F = \mathbb{Z}_2$ because $c_1(\mathbb{C}P^{2n+1}) \equiv 0 \mod 2$. Thus this is consistent with the problem. (If we take $F = \mathbb{Q}$, the eigenvalue is not zero in $\mathbb{Q}$. But Theorem 6.26 shows that the Floer cohomology over $\Lambda_{0,nov}^{\mathbb{Q}}$ vanishes. So the assumption of the problem is not satisfied in this case.) Besides this, we prove this statement for the case of Lagrangian fibers of smooth toric manifolds in [FOOO6]. We do not have any counter example to this statement at the time of writing this paper.

Proof of Theorem 6.26. The set $\pi_2(\mathbb{C}P^{2n+1}, \mathbb{R}P^{2n+1})$ has exactly 2 elements B_1, B_2 satisfying $\mu_{\mathbb{R}P^{2n+1}}(B_1) = \mu_{\mathbb{R}P^{2n+1}}(B_2) = 2n+2$, which is the minimal Maslov number of $\mathbb{R}P^{2n+1}$. By the monotonicity of $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$, a degree counting argument shows that only $\mathcal{M}_2(J; B_1)$ and $\mathcal{M}_2(J; B_2)$, among the moduli spaces $\mathcal{M}_2(J; B)$, $B \in \pi_2(\mathbb{C}P^{2n+1}, \mathbb{R}P^{2n+1})$, contribute to the differential of the spectral sequence. First of all, we note that τ induces an isomorphism modulo orientations

$$\tau_* : \mathcal{M}_2(J; B_1) \longrightarrow \mathcal{M}_2(J; B_2). \tag{6.28}$$

Later we examine whether τ_* preserves the orientation or not, after we specify relative spin structures.

Since $\omega[B_i]$ is the smallest positive symplectic area, $\mathcal{M}_2(J; B_i)$ has no boundary and has a virtual fundamental *cycle* over $\mathbb{Z}$. We also note

$$\dim \mathcal{M}_2(J; B_i) = 2n + 2 + 2n + 1 + 2 - 3 = 2 \dim \mathbb{R}P^{2n+1}.$$

Lemma 6.29. *Consider the evaluation map* $ev : \mathcal{M}_2(J; B_i) \to \mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}$ *for* $i = 1, 2$. *Then we have*

$$ev_*([\mathcal{M}_2(J; B_i)]) = \pm[\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}]$$

that is the fundamental cycle of $\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}$.

Proof. For any distinct two points $p, q \in \mathbb{C}P^{2n+1}$ there exists a holomorphic map $w : S^2 = \mathbb{C} \cup \{\infty\} \to \mathbb{C}P^{2n+1}$ of degree 1 such that $w(0) = p$, $w(1) = q$, which is unique up to the action of $\mathbb{C} \setminus \{0\} \cong \mathrm{Aut}(\mathbb{C}P^1; 0, 1)$. In case $p, q \in \mathbb{R}P^{2n+1}$ the uniqueness implies that $w(\overline{z}) = \tau w(cz)$ for some $c \in \mathbb{C} \setminus \{0\}$. Since $q = w(1) = \tau w(c)$, we have $c = 1$. Namely the restriction of w to the upper or lower half plane defines an element of $\mathcal{M}_2(J; B_1)$, $\mathcal{M}_2(J; B_2)$. Namely there exists $w \in \mathcal{M}_2(J; B_i)$ such that $ev_2(w) = (p, q)$. Conversely, any such element determines a degree one curve by the reflection principle. The proof of Lemma 6.29 is complete. □

We now prove Theorem 6.26 for the case of $\mathbb{R}P^{4n+3}$. In this case, $\mathbb{R}P^{4n+3}$ is τ-relatively spin. Therefore, by Theorem 4.9, the map (6.28) is orientation preserving, because

$$\frac{1}{2}\mu_{\mathbb{R}P^{4n+3}}(B_i) + 2 = 2n + 4$$

is even. Therefore Lemma 6.29 and the definition of the differential d imply

$$d^{4n+3}(PD([p])) = \sum_{i=1}^{2} ev_{0*}\left(\mathcal{M}_2(J; B_i)_{ev_1} \times [p]\right) = \pm 2PD[\mathbb{R}P^{4n+3}],$$

as required.

We next consider the case of $\mathbb{R}P^{4n+1}$. We pick its relative spin structure $[(V,\sigma)]$. By Theorem 4.9 again, the map

$$\tau_* : \mathcal{M}_2(J;B_1)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}_2(J;B_2)^{[(V,\sigma)]}$$

is orientation reversing, because

$$\frac{1}{2}\mu_{\mathbb{R}P^{4n+1}}(B_i) + 2 = 2n+3$$

is odd. On the other hand, by Proposition 3.14 we have $\tau^*[(V,\sigma)] \neq [(V,\sigma)]$. Let $\mathfrak{x}$ be the unique nonzero element of $H^2(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is easy to see that $\mathfrak{x}[B_i] \neq 0$. Then by Proposition 3.10 the identity induces an *orientation reversing* isomorphisms

$$\mathcal{M}_2(J;B_i)^{\tau^*[(V,\sigma)]} \longrightarrow \mathcal{M}_2(J;B_i)^{[(V,\sigma)]}$$

for $i=1,2$. Therefore we can find that

$$\tau_* : \mathcal{M}_2(J;B_1)^{[(V,\sigma)]} \longrightarrow \mathcal{M}_2(J;B_2)^{[(V,\sigma)]}$$

is orientation preserving. The rest of the proof is the same as the case of $\mathbb{R}P^{4n+3}$. The proof of Theorem 6.26 is now complete. □

6.4. **Wall crossing term in** [F2]**.** Let M be a 6-dimensional symplectic manifold and let L be its relatively spin Lagrangian submanifold. Suppose the Maslov index homomorphism $\mu_L : H_2(M,L;\mathbb{Z}) \to \mathbb{Z}$ is zero. In this situation the first named author [F2] introduced an invariant

$$\Psi_J : \mathcal{M}(L;\Lambda_{0,nov}^{\mathbb{C}}) \to \Lambda_{0,nov}^{+\mathbb{C}}.$$

In general, it depends on a compatible almost structure J and the difference $\Psi_J - \Psi_{J'}$ is an element of $\Lambda_{0,nov}^{+\mathbb{Q}}$.

Let us consider the case where $\tau : M \to M$ is an anti-symplectic involution and $L = \mathrm{Fix}\,\tau$. We take the compatible almost complex structures J_0, J_1 such that $\tau_* J_0 = -J_0$, $\tau_* J_1 = -J_1$. Moreover, we assume that there exists a one parameter family of compatible almost complex structures $\mathcal{J} = \{J_t \mid t \in [0,1]\}$ such that $\tau_* J_t = -J_t$. Then

$$\Psi_{J_0} = \Psi_{J_1}. \tag{6.29}$$

Namely, wall crossing does not occur in this situation. We can prove (6.29) (which is also mentioned in Subsection 8.3 [F2]) by the method of this paper as follows.

Let $\alpha \in H_2(M;\mathbb{Z})$. Denote by $\mathcal{M}_1(\alpha;J)$ the moduli space of J-holomorphic *sphere* with one interior marked point and of homology class α. We have an evaluation map $ev : \mathcal{M}_1(\alpha;J) \to M$. We assume

$$ev(\mathcal{M}_1(\alpha;J_0)) \cap L = ev(\mathcal{M}_1(\alpha;J_1)) \cap L = \emptyset \tag{6.30}$$

for any $\alpha \neq 0$. Since the virtual dimension of $\mathcal{M}_1(\alpha;J)$ is 2, (6.30) holds in generic cases. The space

$$\mathcal{M}_1(\alpha;\mathcal{J};L) = \bigcup_{t\in[0,1]} \{t\} \times (\mathcal{M}_1(\alpha;J_t)_{ev} \times_M L) \tag{6.31}$$

has a Kuranishi structure of dimension 0. The assumption (6.30) implies that (6.31) has no boundary. Therefore its virtual fundamental cycle is well-defined and gives

a rational number, which we denote by $\#\mathcal{M}_1(\alpha; \mathcal{J}; L)$. By Theorem 1.5 [F2] we have

$$\Psi_{J_1} - \Psi_{J_0} = \sum_{\alpha} \#\mathcal{M}_1(\alpha; \mathcal{J}; L) T^{\omega(\alpha)}.$$

The next lemma implies that this number is zero in our situation. The involution naturally induces a map $\tau : \mathcal{M}_1(\alpha; \mathcal{J}; L) \to \mathcal{M}_1(\tau_*\alpha; \mathcal{J}; L)$.

Lemma 6.30. *The map $\tau : \mathcal{M}_1(\alpha; \mathcal{J}; L) \to \mathcal{M}_1(\tau_*\alpha; \mathcal{J}; L)$ is orientation reversing.*

Proof. In the same way as Proposition 4.8, we can prove that $\tau : \mathcal{M}_1(\alpha; J) \to \mathcal{M}_1(\tau_*\alpha; J)$ is orientation reversing. In fact, this case is similar to the case $k = -1$, $\mu_L(\beta) = 2c_1(\alpha) = 0$ and $m = 1$ of Proposition 4.8. The lemma follows immediately. $\square$

7. Appendix: Review of Kuranishi structure – orientation and group action

In this appendix, we briefly review the orientation on a space with Kuranishi structure and notion of group action on a space with Kuranishi structure for the readers convenience. For more detailed explanation, we refer Sections A1.1 [FOOO3] and A1.3 [FOOO3] for Subsections 7.1 and 7.2, respectively.

7.1. **Orientation.**

To define orientation on a space with Kuranishi structure, we first recall the notion of tangent bundle of it. Let $\mathcal{M}$ be a compact topological space and let $\mathcal{M}$ have a Kuranishi structure. That is, $\mathcal{M}$ has a collection of a finite number of Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p), p \in \mathcal{M}$ such that

- (k-1) V_p is a finite dimensional smooth manifold which may have boundaries or corners;
- (k-2) E_p is a finite dimensional real vector space and $\dim V_p - \dim E_p$ is independent of p;
- (k-3) Γ_α is a finite group acting smoothly and effectively on V_p, and on E_p linearly;
- (k-4) s_p, which is called a *Kuranishi map*, is a Γ_p- equivariant smooth section of the vector bundle $E_p \times V_p \to V_p$ called an *obstruction bundle*;
- (k-5) $\psi_p : s_p^{-1}(0)/\Gamma_p \to \mathcal{M}$ is a homeomorphism to its image;
- (k-6) $\bigcup_p \psi_p(s_p^{-1}(0)/\Gamma_p) = \mathcal{M}$;
- (k-7) the collection $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p \in \mathcal{M}}$ satisfies certain compatibility conditions under coordinate change.

See Definition A1.3 and Definition A1.5 in [FOOO3] for the precise definition and description of the *coordinate change* and the compatibility conditions in (k-7), respectively. We denote by $\mathcal{P}$ the finite set of $p \in \mathcal{M}$ above. By Lemma 6.3 [FO], we may assume that $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p \in P}$ is a *good coordinate system* in the sense of Definition 6.1 in [FO]. In other words, there is a partial order $<$ on $\mathcal{P}$ such that the following conditions hold: Let $q < p, (p, q \in \mathcal{P})$ with $\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \neq \emptyset$. Then there exist

- (gc-1) a Γ_q-invariant open subset V_{pq} of V_q such that
$$\psi_q^{-1}(\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q)) \subset V_{pq}/\Gamma_q,$$
- (gc-2) an injective group homomorphism $h_{pq} : \Gamma_q \to \Gamma_p$,

(gc-3) an h_{pq}-equivariant smooth embedding $\phi_{pq} : V_{pq} \to V_p$ such that the induced map $V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is injective,

(gc-4) an h_{pq}- equivariant embedding $\widehat{\phi}_{pq} : E_q \times V_{pq} \to E_p \times V_p$ of vector bundles which covers ϕ_{pq} and satisfies

$$\widehat{\phi}_{pq} \circ s_q = s_p \circ \phi_{pq}, \quad \psi_q = \psi_p \circ \underline{\phi}_{pq}.$$

Here $\underline{\phi}_{pq} : V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is the map induced by ϕ_{pq}.

Moreover, if $r < q < p$ and $\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \cap \psi_r(s_r^{-1}(0)/\Gamma_r) \neq \emptyset$, then there exists

(gc-5) $\gamma_{pqr} \in \Gamma_p$ such that

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr}, \quad \widehat{\phi}_{pq} \circ \widehat{\phi}_{qr} = \gamma_{pqr} \cdot \widehat{\phi}_{pr}.$$

Here the second and third equalities hold on $\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$ and on $E_r \times (\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}))$, respectively.

Now we identify a neighborhood of $\phi_{pq}(V_{pq})$ in V_p with a neighborhood of the zero section of the normal bundle $N_{V_{pq}}V_p \to V_{pq}$. Then the differential of the Kuranishi map s_p along the fiber direction defines an h_{pq}-equivariant bundle homomorphism

$$d_{\text{fiber}}s_p : N_{V_{pq}}V_p \to E_p \times V_{pq}.$$

Definition 7.1. We say that the space $\mathcal{M}$ with Kuranishi structure *has a tangent bundle* if $d_{\text{fiber}}s_p$ induces a bundle isomorphism

$$N_{V_{pq}}V_p \cong \frac{E_p \times V_{pq}}{\widehat{\phi}_{pq}(E_q \times V_{pq})} \tag{7.1}$$

as Γ_q-equivariant bundles on V_{pq}.

Definition 7.2. Let $\mathcal{M}$ be a space with Kuranishi structure with a tangent bundle. We say that the Kuranishi structure on $\mathcal{M}$ is *orientable* if there is a trivialization of

$$\Lambda^{\text{top}}E_p^* \otimes \Lambda^{\text{top}}TV_p$$

which is compatible with the isomorphism (7.1) and whose homotopy class is preserved by the Γ_p-action. The *orientation* is the choice of the homotopy class of such a trivialization.

Pick such a trivialization. Then we define an orientation on the zero locus $s_p^{-1}(0)$ of the Kuranishi map s_p, which may be assumed so that $p \in s_p^{-1}(0)$, by the following equation:

$$E_p \times T_p s_p^{-1}(0) = T_p V.$$

Since we pick a trivialization of $\Lambda^{\text{top}}E_p^* \otimes \Lambda^{\text{top}}TV_p$ as in Definition 7.2, the above equality determines an orientation on $s_p^{-1}(0)$, and also on $s_p^{-1}(0)/\Gamma_p$. See Section 8.2 [FOOO3] for more detailed explanation of orientation on a space with Kuranishi structure.

7.2. **Group action.** We next recall the definitions of a finite group action on a space with Kuranishi structure and its quotient space. In this paper we used the $\mathbb{Z}_2$-action and its quotient space (in the proof of Theorem 1.5).

Let $\mathcal{M}$ be a compact topological space with Kuranishi structure. We first define the notion of automorphism of Kuranishi structure.

Definition 7.3. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a homeomorphism of $\mathcal{M}$. We say that it induces an *automorphism of Kuranishi structure* if the following holds: Let $p \in \mathcal{M}$ and $p' = \varphi(p)$. Then, for the Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(V_{p'}, E_{p'}, \Gamma_{p'}, \psi_{p'}, s_{p'})$ of p and p' respectively, there exist $\rho_p : \Gamma_p \to \Gamma_{p'}$, $\varphi_p : V_p \to V_{p'}$, and $\widehat{\varphi}_p : E_p \to E_{p'}$ such that

(au-1) ρ_p is an isomorphism of groups;
(au-2) φ_p is a ρ_p-equivariant diffeomorphism;
(au-3) $\widehat{\varphi}_p$ is a ρ_p-equivariant bundle isomorphism which covers φ_p;
(au-4) $s_{p'} \circ \varphi_p = \widehat{\varphi}_p \circ s_p$;
(au-5) $\psi_{p'} \circ \underline{\varphi}_p = \varphi \circ \psi_p$, where $\underline{\varphi}_p : s_p^{-1}(0)/\Gamma_p \to s_{p'}^{-1}(0)/\Gamma_{p'}$ is a homeomorphism induced by $\varphi_p|_{s_p^{-1}(0)}$.

We require that ρ_p, φ_p, $\widehat{\varphi}_p$ above satisfy the following compatibility conditions with the coordinate changes of Kuranishi structure: Let $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ and $q' \in \psi_{p'}(s_{p'}^{-1}(0)/\Gamma_{p'})$ such that $\varphi(q) = q'$. Let $(\widehat{\phi}_{pq}, \phi_{pq}, h_{pq})$, $(\widehat{\phi}_{p'q'}, \phi_{p'q'}, h_{p'q'})$ be the coordinate changes. Then there exists $\gamma_{pqp'q'} \in \Gamma_{p'}$ such that the following conditions hold:

(auc-1) $\rho_p \circ h_{pq} = \gamma_{pqp'q'} \cdot (h_{p'q'} \circ \rho_q) \cdot \gamma_{pqp'q'}^{-1}$;
(auc-2) $\varphi_p \circ \phi_{pq} = \gamma_{pqp'q'} \cdot (\phi_{p'q'} \circ \varphi_q)$;
(auc-3) $\widehat{\varphi}_p \circ \widehat{\phi}_{pq} = \gamma_{pqp'q'} \cdot (\widehat{\phi}_{p'q'} \circ \widehat{\varphi}_q)$.

Then we call $((\rho_p, \varphi_p, \widehat{\varphi}_p)_p; \varphi)$ an *automorphism of the Kuranishi structure*.

Definition 7.4. We say that an automorphism $((\rho_p, \varphi_p, \widehat{\varphi}_p)_p; \varphi)$ is *conjugate* to $((\rho'_p, \varphi'_p, \widehat{\varphi}'_p)_p; \varphi')$, if $\varphi = \varphi'$ and if there exists $\gamma_p \in \Gamma_{\varphi(p)}$ for each p such that

(cj-1) $\rho'_p = \gamma_p \cdot \rho_p \cdot \gamma_p^{-1}$;
(cj-2) $\varphi'_p = \gamma_p \cdot \varphi_p$;
(cj-3) $\widehat{\varphi}'_p = \gamma_p \cdot \widehat{\varphi}_p$.

The *composition* of the two automorphisms is defined by the following formula:

$$\begin{aligned}&((\rho^1_p, \varphi^1_p, \widehat{\varphi}^1_p)_p; \varphi^1) \circ ((\rho^2_p, \varphi^2_p, \widehat{\varphi}^2_p)_p; \varphi^2)\\ &= ((\rho^1_{\varphi^2(p)} \circ \rho^2_p, \varphi^1_{\varphi^2(p)} \circ \varphi^2_p, \widehat{\varphi}^1_{\varphi^2(p)} \circ \widehat{\varphi}^2_p)_p; \varphi^1 \circ \varphi^2).\end{aligned}$$

Then we can easily check that the right hand side also satisfies the compatibility conditions (auc-1)-(au-3). Moreover, we can find that the composition induces the composition of the conjugacy classes of automorphisms.

We say that the *lift is orientation preserving* if it is compatible with the trivialization of $\Lambda^{\rm top} E_p^* \otimes \Lambda^{\rm top} TV_p$.

Let $\mathrm{Aut}(\mathcal{M})$ be the set of all conjugacy classes of the automorphisms of $\mathcal{M}$ and let $\mathrm{Aut}_0(\mathcal{M})$ be the set of all conjugacy classes of the orientation preserving automorphisms of $\mathcal{M}$. Both of them become groups by composition.

Definition 7.5. Let G be a finite group which acts on a compact space $\mathcal{M}$. Assume that $\mathcal{M}$ has a Kuranishi structure. We say that G *acts* on $\mathcal{M}$ (as a space with Kuranishi structure) if, for each $g \in G$, the homeomorphism $x \mapsto gx$, $X \to X$ is

lifted to an automorphism g_* of the Kuranishi structure and the composition of g_* and h_* is conjugate to $(gh)_*$. In other words, an action of G to X is a group homomorphism $G \to \mathrm{Aut}(X)$.

An *involution* of a space with Kuranishi structure is a $\mathbb{Z}_2$ action.

Then we can show the following:

Lemma 7.6 (Lemma A1.49 [FOOO3]). *If a finite group G acts on a space $\mathcal{M}$ with Kuranishi structure, then the quotient space $\mathcal{M}/G$ has a Kuranishi structure.*

If $\mathcal{M}$ has a tangent bundle and the action preserves it, then the quotient space has a tangent bundle. If $\mathcal{M}$ is oriented and the action preserves the orientation, then the quotient space has an orientation.

References

[A] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. 1 (2007), 51–91, arXiv:0706.3207v2.

[CMS] R. Castaño-Bernard, D. Matessi and J. Solomon, *Symmetries of Lagrangian fibrations*, preprint, arXiv:0908.0966.

[C] C.-H. Cho, *Holomorphic discs, spin structures and Floer cohomology of the Clifford tori*, Internat. Math. Res. Notices 35 (2004), 613–640.

[F1] K. Fukaya, *Morse homotopy and its quantization*, in 'Geometric topology (Athens, GA, 1993)', ed. W. Kazez, 409–440, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.

[F2] K. Fukaya, *Counting pseudo-holomorphic discs in Calabi-Yau 3 fold*, submitted, arXiv:0908.0148.

[FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction,* Kyoto University preprint, 2000.

[FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction,* expanded version of [FOOO1], 2006 & 2007.

[FOOO3] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction,* vol. 46-1 & 46-2, AMS/IP Studies in Advanced Math., American Math. Soc. and International Press. (2009).

[FOOO4] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J., 151 (2010), 23–175, arXiv:0802.1703.

[FOOO5] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds II: Bulk deformations*, submitted, arXiv:0810.5774.

[FOOO6] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds I*, in preparation.

[FO] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933–1048.

[Oh1] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo- holomorphic disks I,* & *II* Comm. Pure and Appl. Math. 46 (1993), 949–994 & 995–1012. Addenda, ibid, 48 (1995), 1299–1302.

[Oh2] Y.-G. Oh *Floer cohomology, spectral sequence and the Maslov class of Lagrangian embeddings*, Internat. Math. Res. Notices 7 (1996), 305–346.

[Oht] H. Ohta, *Obstruction to and deformation of Lagrangian intersection Floer cohomology*, Symplectic Geometry and Mirror Symmetry (Seoul, 2000), K. Fukaya, Y.-G. Oh, K. Ono and G. Tian ed., World Sci. Publishing, River Edge, 2001, pp. 281–309.

[PSW] R. Pandharipande, J. Solomon and J. Walcher, *Disk enumeration on the quintic 3-fold*, J. Amer. Math. Soc. 21 (2008), no. 4, 1169–1209.

[Si] V. de Silva, *Products on symplectic Floer homology*, Thesis, Oxford Uinv. (1997).

[So] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions,* preprint, arXiv:math/0606429.

[Wa] J. Walcher, *Opening mirror symmetry on the quintic*, Commun. Math. Phys. 276 (2007), 671–689.

[We] J-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. 162 (2005), no. 1, 195–234.

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